Math 173 - Final Exam

May 15, 2017

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (15 points) Let P and Q be the points $(2, 1, -3)$ and $(4, -2, 7)$, respectively.

   (a) Find a vector of length 2 in the direction of $\overrightarrow{PQ}$.

$$\overrightarrow{PQ} = (4-2)\hat{\imath} + (-2-1)\hat{\jmath} + (7-(-3))\hat{k}$$
$$= 2\hat{\imath} - 3\hat{\jmath} + 10\hat{k}$$
$$\|\overrightarrow{PQ}\| = \sqrt{4+9+100} = \sqrt{113}$$
$$\boxed{\frac{2}{\sqrt{113}}\overrightarrow{PQ} = \frac{4}{\sqrt{113}}\hat{\imath} - \frac{6}{\sqrt{113}}\hat{\jmath} + \frac{20}{\sqrt{113}}\hat{k}}$$

   (b) Find a unit vector in the xy-plane that is orthogonal to $\overrightarrow{PQ}$.

$$\vec{v} = 3\hat{\imath} + 2\hat{\jmath} \Rightarrow \overrightarrow{PQ}\cdot\vec{v} = 6-6+0 = 0$$
$$\|\vec{v}\| = \sqrt{9+4} = \sqrt{13}$$
$$\boxed{\frac{1}{\sqrt{13}}(3\hat{\imath} + 2\hat{\jmath})}$$

   (c) Find the projection of $\overrightarrow{PQ}$ onto $\underbrace{3\hat{\imath} + 2\hat{\jmath} + \hat{k}}_{\vec{u}}$.

$$\vec{u}\cdot\overrightarrow{PQ} = 10$$
$$\vec{u}\cdot\vec{u} = 9+4+1 = 14$$
$$\text{proj}_{\vec{u}}\overrightarrow{PQ} = \frac{\vec{u}\cdot\overrightarrow{PQ}}{\vec{u}\cdot\vec{u}}\vec{u} = \frac{10}{14}\vec{u} = \boxed{\frac{15}{7}\hat{\imath} + \frac{10}{7}\hat{\jmath} + \frac{5}{7}\hat{k}}$$

2. (10 points) Consider the plane described by the equation $5x - 3y - 2z = 8$.

(a) Find an equation of the parallel plane passing through the point $(2, 3, 5)$.

$$5x - 3y - 2z = 5(2) - 3(3) - 2(5)$$
$$= -9$$

$$\boxed{5x - 3y - 2z = -9}$$

(b) Find the angle between the given plane and the plane with equation $2x + 7y = 12$.

$$\vec{n}_1 = 5\hat{i} - 3\hat{j} - 2\hat{k} \qquad \vec{n}_2 = 2\hat{i} + 7\hat{j}$$

$$\cos\theta = \left| \frac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\| \, \|\vec{n}_2\|} \right| = \left| \frac{5(2) - 3(7) - 2(0)}{\sqrt{25+9+4}\,\sqrt{4+49}} \right|$$

$$= \frac{11}{\sqrt{38}\,\sqrt{53}} \Rightarrow \theta = \cos^{-1}\left(\frac{11}{\sqrt{38}\,\sqrt{53}}\right) \approx 1.323$$
$$\approx 75.8^\circ$$

3. (10 points) A plane passes through the points $P\,(2, 1, 3)$, $Q\,(-7, 6, -1)$ and $R\,(3, 0, -1)$. Find a set of parametric equations for the line normal to the plane and passing through $(2, 1, 3)$.

$$\vec{PQ} = (-7-2)\hat{i} + (6-1)\hat{j} + (-1-3)\hat{k} = -9\hat{i} + 5\hat{j} - 4\hat{k}$$
$$\vec{PR} = (3-2)\hat{i} + (0-1)\hat{j} + (-1-3)\hat{k} = \hat{i} - \hat{j} - 4\hat{k}$$

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -9 & 5 & -4 \\ 1 & -1 & -4 \end{vmatrix}$$

$$= \hat{i}(-24) - \hat{j}(40) + \hat{k}(4)$$

I'll use $\vec{n} = 6\hat{i} + 10\hat{j} - \hat{k}$

$$\boxed{\begin{aligned} x &= 6t + 2 \\ y &= 10t + 1 \\ z &= -t + 3 \end{aligned}}$$

4. (10 points) The quarterback of a football team throws the ball with an initial speed of 54 feet per second, at an angle of 36°, and at a height of 6 feet above the playing field. How far downfield has the ball traveled at the moment when it reaches its maximum height? (Ignore air resistance and use $g = 32\,\text{ft}/\text{sec}^2$.)

$$\vec{r}(t) = 54\cos 36^\circ\, t\,\hat{i} + \left(-16t^2 + 54\sin 36^\circ\, t + 6\right)\hat{j}$$

$$-32t + 54\sin 36^\circ = 0$$

$$\Rightarrow \quad t = \frac{54\sin 36^\circ}{32} \approx 0.992$$

$$54\cos 36^\circ\left(\frac{54\sin 36^\circ}{32}\right) \approx \boxed{43.3 \text{ FT}}$$

5. (15 points) Let $\vec{r}(t) = 3t\hat{\imath} + \sin 2t\,\hat{\jmath} + \cos 2t\,\hat{k}$. Find (a) the unit tangent vector, (b) the principal unit normal vector, and (c) the length of the graph from $t = 0$ to $t = \pi/2$.

$$\vec{r}\,'(t) = 3\hat{i} + 2\cos 2t\,\hat{j} - 2\sin 2t\,\hat{k}$$

$$\|\vec{r}\,'(t)\| = \sqrt{9+4} = \sqrt{13}$$

a) $$\boxed{\hat{T}(t) = \frac{1}{\sqrt{13}}\left(3\hat{i} + 2\cos 2t\,\hat{j} - 2\sin 2t\,\hat{k}\right)}$$

$$\hat{T}'(t) = \frac{1}{\sqrt{13}}\left(-4\sin 2t\,\hat{j} - 4\cos 2t\,\hat{k}\right)$$

b) $$\boxed{\hat{N}(t) = -\sin 2t\,\hat{j} - \cos 2t\,\hat{k}}$$

(c) $$\int_0^{\pi/2} \sqrt{13}\, dt = \boxed{\frac{\sqrt{13}\,\pi}{2}}$$

6. (10 points) Suppose that $w = 3xy + yz$ and that x, y, and z are functions of u and v such that

$$x = \ln u + \cos v, \qquad y = 1 + u \sin v, \qquad z = uv.$$

Use the chain rule to find $\partial w/\partial u$ at $(u, v) = (1, \pi)$.

At $(1, \pi)$,
$x = -1$,
$y = 1$
$z = \pi$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$= (3y)\left(\frac{1}{u}\right) + (3x+z)(\sin v) + (y)(v)$$

$$\left.\frac{\partial w}{\partial u}\right|_{(u,v)=(1,\pi)} = 3(1) + (\pi - 3)(0) + \pi = \boxed{3+\pi}$$

7. (10 points) The temperature at the point (x, y) on a ceramic plate is given by

$$T(x, y) = 12 + \cos(\pi xy) + x^2 + xy, \quad -2 \le x \le 2, \; -2 \le y \le 2.$$

(a) Find the directional derivative of T at the point $(1, 1)$ in the direction of $\underbrace{3\hat{\imath} + \hat{\jmath}}_{}$.

$$\frac{\vec{u}}{\|\vec{u}\|} = \frac{3\hat{\imath} + \hat{\jmath}}{\sqrt{10}}$$

$$\vec{\nabla} T(x,y) = (-\pi y \sin \pi xy + 2x + y)\,\hat{\imath} + (-\pi x \sin \pi xy + x)\,\hat{\jmath}$$

$$\vec{\nabla} T(1,1) = 3\hat{\imath} + \hat{\jmath}$$

$$\vec{\nabla} T(1,1) \cdot \frac{\vec{u}}{\|\vec{u}\|} = \sqrt{10}$$

(b) Find the direction of greatest temperature decrease at the point $(1, 1)$.

$$\boxed{-\vec{\nabla} T(1,1) = -3\hat{\imath} - \hat{\jmath}}$$

OPPOSITE DIRECTION OF GRADIENT VECTOR.

8. (10 points) Consider the surface in 3-space defined by the equation $\overbrace{x^2yz^3}^{F(x,y,z)} = 2$. Find an equation of the plane tangent to the surface at the point $(1, 2, 1)$.

$$\vec{\nabla} F(x,y,z) = 2xyz^3\,\hat{i} + x^2z^3\,\hat{j} + 3x^2yz^2\,\hat{k}$$

$$\vec{n} = \vec{\nabla} F(1,2,1) = 4\hat{i} + \hat{j} + 6\hat{k}$$

$$4(x-1) + (y-2) + 6(z-1) = 0$$

- OR -

$$\boxed{4x + y + 6z = 12}$$

9. (10 points) Use Lagrange multipliers to find the minimum and maximum values of $f(x, y, z) = 2x + 3y + z$ on the sphere $\underbrace{x^2 + y^2 + z^2}_{g(x,y,z)} = 1$.

$$\vec{\nabla} f = \lambda \vec{\nabla} g \qquad g = 1 \quad \Rightarrow \quad \begin{cases} 2 = \lambda 2x \\ 3 = \lambda 2y \\ 1 = \lambda 2z \\ x^2+y^2+z^2 = 1 \end{cases} \longrightarrow \quad x = \frac{1}{\lambda} \quad y = \frac{3}{2\lambda} \quad z = \frac{1}{2\lambda}$$

$$\frac{1}{\lambda^2} + \frac{9}{4\lambda^2} + \frac{1}{4\lambda^2} = 1$$

$$\lambda^2 = \frac{14}{4}$$

$$\lambda = \pm \frac{\sqrt{14}}{2}$$

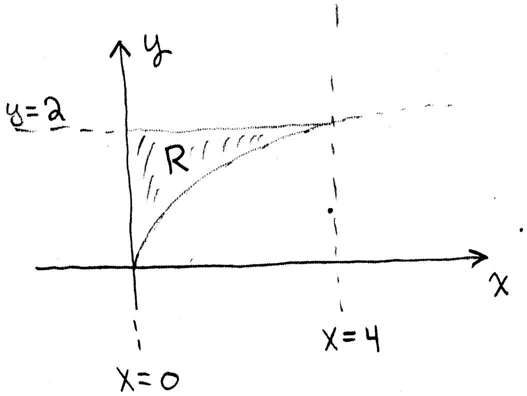
$$x = \pm \frac{2}{\sqrt{14}} \qquad y = \pm \frac{3}{\sqrt{14}} \qquad z = \pm \frac{1}{\sqrt{14}}$$

$$f\left(\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right) = \frac{14}{\sqrt{14}} = \sqrt{14} = \text{MAX}$$

$$f\left(-\frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}}, -\frac{1}{\sqrt{14}}\right) = -\sqrt{14} = \text{MIN}$$

10. (12 points) Sketch the region of integration, reverse the order of integration, and **evaluate the iterated integral by hand.**

$$\int_0^4 \int_{\sqrt{x}}^2 \frac{x}{y^5+1}\,dy\,dx$$



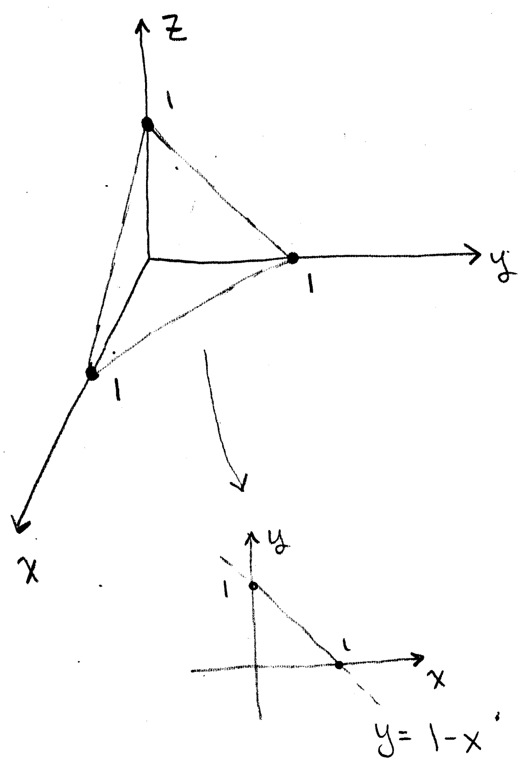
$$= \int_{y=0}^{y=2} \int_{x=0}^{x=y^2} \frac{x}{y^5+1}\,dx\,dy$$

$$= \frac{1}{2}\int_0^2 \frac{y^4}{y^5+1}\,dy \qquad u = y^5+1, \quad du = 5y^4\,dy$$

$$= \frac{1}{10}\int_1^{33} \frac{1}{u}\,du = \boxed{\frac{1}{10}\ln 33}$$

≈ 0.34965

11. (12 points) Use a triple integral to find the volume of the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$. (Evaluate the integral by hand.)



$$\text{Volume} = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} (1-x-y)\,dy\,dx$$

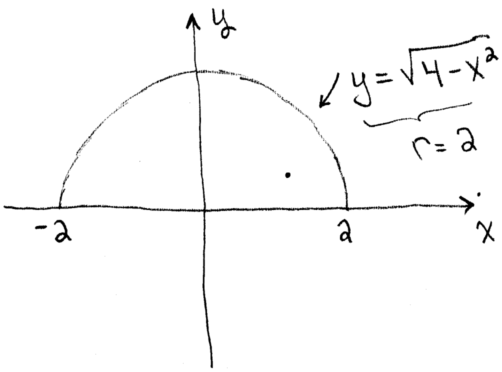
$$= \int_0^1 \left[(1-x) - x(1-x) - \frac{1}{2}(1-x)^2\right] dx$$

$$= 1 - \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{2} + \frac{1}{2} - \frac{1}{6}$$

$$= \frac{1}{3} - \frac{1}{6} = \frac{1}{6}$$

12. (13 points) Convert to cylindrical coordinates and evaluate by hand.

$$\int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} \int_{0}^{3} \frac{1}{1+x^2+y^2}\, dz\, dy\, dx$$



$$\int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=2} \int_{z=0}^{z=3} \frac{r}{1+r^2}\, dz\, dr\, d\theta$$

$$= 3\pi \int_0^2 \frac{r}{1+r^2}\, dr \qquad \begin{aligned} u &= 1+r^2 \\ du &= 2r\,dr \end{aligned}$$

$$= \frac{3\pi}{2} \ln(1+r^2)\Big|_0^2 = \boxed{\frac{3\pi \ln 5}{2}}$$

≈ 7.5843

13. (13 points) Show that $\vec{F}(x,y) = \frac{1}{2}xy\hat{\imath} + \frac{1}{4}x^2\hat{\jmath}$ is a conservative vector field. Then use any method to evaluate $\int_C \vec{F} \cdot d\vec{r}$, where C is any smooth curve from $(0,0)$ to $(1,1)$. Evaluate your integral by hand.

Suppose $\vec{\nabla} f = \vec{F}$...

$$\frac{\partial f}{\partial x} = \frac{1}{2}xy \Rightarrow f(x,y) = \frac{1}{4}x^2y + g(y)$$

$$\frac{\partial f}{\partial y} = \frac{1}{4}x^2 \Rightarrow f(x,y) = \frac{1}{4}x^2y + h(x)$$

WE'LL TAKE $\boxed{f(x,y) = \frac{1}{4}x^2y}$

THIS SHOWS $\vec{F}$ IS CONSERVATIVE!

$$\int_C \vec{F}\cdot d\vec{r} = f(1,1) - f(0,0)$$

$$= \boxed{\frac{1}{4}}$$