Math 173 - Test 2

March 21, 2019

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (4 points) Let $\vec{r}(t) = 2\sin 5t\,\hat{\imath} + 8\hat{\jmath} - 2\cos 5t\,\hat{k}$. Compute $\|\vec{r}(t)\|$ and then briefly explain (without computing) why you would expect that $\vec{r}(t)\cdot\vec{r}'(t) = 0$.

$$\|\vec{r}(t)\| = \sqrt{4\sin^2 5t + 64 + 4\cos^2 5t}$$

$$= \sqrt{4+64} = \sqrt{68}$$

SINCE $\vec{r}$ HAS CONSTANT MAGNITUDE, $\vec{r}(t)$ IS ORTHOGONAL TO $\vec{r}'(t)$.

2. (6 points) Determine $\vec{r}(t)$ if $\dfrac{d\vec{r}}{dt} = \dfrac{1}{1+t^2}\hat{\imath} + \dfrac{1}{t^2}\hat{\jmath} + \dfrac{1}{t}\hat{k}$ and $\vec{r}(1) = 2\hat{\imath}$.

INTEGRATE ...

$$\vec{r}(t) = \left(\tan^{-1} t + c_1\right)\hat{\imath} + \left(-\frac{1}{t} + c_2\right)\hat{\jmath} + \left(\ln|t| + c_3\right)\hat{k}$$

$$\vec{r}(1) = \left(\frac{\pi}{4} + c_1\right)\hat{\imath} + \left(-1 + c_2\right)\hat{\jmath} + c_3\hat{k} = 2\hat{\imath}$$

$$\Rightarrow c_1 = 2 - \frac{\pi}{4},\quad c_2 = 1,\quad c_3 = 0$$

$$\vec{r}(t) = \left(\tan^{-1} t + 2 - \frac{\pi}{4}\right)\hat{\imath} + \left(1 - \frac{1}{t}\right)\hat{\jmath} + \left(\ln|t|\right)\hat{k}$$

3. (4 points) A object is falling in such a way that its position at time t is given by

$$\vec{r}(t) = 171\sqrt{3}\,t\,\hat{\imath} + (-4.9t^2 + 171t + 2)\,\hat{\jmath}.$$

What is the maximum height of the object?

$$-9.8t + 171 = 0$$

$$\Rightarrow t = \frac{171}{9.8}$$

$$-4.9\left(\frac{171}{9.8}\right)^2 + 171\left(\frac{171}{9.8}\right) + 2 \approx \boxed{1493.9}$$

$\hat{\imath}$ - comp is 1500 when $\hat{\jmath}$ - comp is 5.

4. (8 points) A projectile is fired from 5 feet above the ground with an initial speed of 2800 ft/s. The projectile is supposed to hit a target 1500 feet away and 5 feet above the ground. Find the initial angle. (Ignore all forces except gravity and use $g = 32\,\text{ft/s}^2$.)

$$\vec{r}(t) = 2800 \cos\theta\, t\, \hat{\imath} + (-16t^2 + 2800 \sin\theta\, t + 5)\, \hat{\jmath}$$

$$2800 \cos\theta\, t = 1500$$

$$\underbrace{-16t^2 + 2800 \sin\theta\, t = 0}$$

$$t = 0 \quad \text{or} \quad t = \frac{2800 \sin\theta}{16}$$

$$(2800\cos\theta)\left(\frac{2800 \sin\theta}{16}\right) = 1500$$

$$2 \cos\theta \sin\theta = \frac{1500 \times 16}{1400 \times 2800}$$

$$\sin 2\theta = 24000/3920000.$$

$$2\theta \approx 0.35079°$$

$$\theta \approx 0.17540°$$

5. (2 points) If an object is moving along a straight line, what can be said about the principal unit normal vector?

$\hat{T}(t)$ is constant.

$\hat{T}'(t) = 0 \Rightarrow \boxed{\hat{N}(t) \text{ does not exist.}}$

6. (6 points) For a smooth curve described by $\vec{r}(t)$, the following facts are known:

- $\hat{T}(t_0) = \frac{3}{5}\hat{\imath} - \frac{4}{5}\hat{k}$
- $\hat{T}'(t_0) = -4\hat{\imath} - 3\hat{k}$
- $\|\vec{r}'(t_0)\| = 10$

(a) Find $\vec{r}'(t_0)$.

$$\hat{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} \Rightarrow \vec{r}'(t_0) = 10\left(\frac{3}{5}\hat{\imath} - \frac{4}{5}\hat{k}\right) = \boxed{6\hat{\imath} - 8\hat{k}}$$

(b) Find $\hat{N}(t_0)$.

$$\hat{N}(t) = \frac{\hat{T}'(t)}{\|\hat{T}'(t)\|}, \quad \|\hat{T}'(t_0)\| = \sqrt{16+9} = 5.$$

$$\Rightarrow \hat{N}(t_0) = \boxed{-\frac{4}{5}\hat{\imath} - \frac{3}{5}\hat{k}}$$

7. (8 points) Determine $\hat{N}(0)$ if $\vec{r}(t) = \sqrt{2}\,t\,\hat{\imath} + e^t\,\hat{\jmath} + e^{-t}\,\hat{k}$.
(Hint: It may be helpful to recognize that $2 + e^{2t} + e^{-2t} = (e^t + e^{-t})^2$.)

$$\vec{r}'(t) = \sqrt{2}\,\hat{\imath} + e^t\,\hat{\jmath} - e^{-t}\,\hat{k}$$

$$\|\vec{r}'(t)\| = \underbrace{\sqrt{2 + e^{2t} + e^{-2t}}}_{\text{HINT.}} = (e^t + e^{-t})$$

$$\hat{T}(t) = \frac{\sqrt{2}\,\hat{\imath} + e^t\,\hat{\jmath} - e^{-t}\,\hat{k}}{e^t + e^{-t}}$$

$$\hat{T}'(t) = \frac{(e^t + e^{-t})(e^t\,\hat{\jmath} + e^{-t}\,\hat{k}) - (\sqrt{2}\,\hat{\imath} + e^t\,\hat{\jmath} - e^{-t}\,\hat{k})(e^t - e^{-t})}{(e^t + e^{-t})^2}$$

$$\hat{T}'(0) = \frac{2(\hat{\jmath} + \hat{k}) - (\sqrt{2}\,\hat{\imath} + \hat{\jmath} - \hat{k})(0)}{4}$$

$$= \frac{1}{2}(\hat{\jmath} + \hat{k})$$

$$\hat{N}(0) = \frac{\hat{T}'(0)}{\|\hat{T}'(0)\|} = \frac{1}{\sqrt{2}}(\hat{\jmath} + \hat{k})$$

8. (5 points) An object spirals around the x-axis on the space curve described by $\vec{r}(t) = 7t\,\hat{\imath} + 2\sin t\,\hat{\jmath} + 2\cos t\,\hat{k}$. Determine the length of the path from $t = 0$ to $t = \pi$.

$$\vec{r}'(t) = 7\hat{\imath} + 2\cos t\,\hat{\jmath} - 2\sin t\,\hat{k}$$

$$\|\vec{r}'(t)\| = \sqrt{49 + 4\cos^2 t + 4\sin^2 t}$$

$$= \sqrt{53}$$

$$\int_0^{\pi} \sqrt{53}\,dt = \boxed{\sqrt{53}\,\pi}$$

9. (9 points) Let $z = f(x, y) = 2x^2 - xy$.

(a) Compute the total differential dz.

$$dz = f_x\,dx + f_y\,dy$$

$$dz = (4x - y)\,dx + (-x)\,dy$$

(b) Use differentials to approximate Δz if (x, y) changes from $(1, 1)$ to $(0.98, 1.03)$.

$\Delta x = -0.02$

$\Delta y = 0.03$

$$\Delta z \approx f_x(1,1)\,\Delta x + f_y(1,1)\,\Delta y$$

$$\Delta z \approx [4(1) - 1](-0.02) + [-1](0.03) = \boxed{-0.09}$$

(c) Referring to part (b), compute the exact value of Δz.

$$\Delta z = f(0.98, 1.03) - f(1,1)$$

$$= 0.9114 - 1 = -0.0886$$

10. (5 points) Determine the curvature as a function of x for the plane curve described by $y = \ln x$.

$$K = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}} \qquad \frac{dy}{dx} = \frac{1}{x} \qquad \frac{d^2y}{dx^2} = -\frac{1}{x^2}$$

$$K = \frac{\frac{1}{x^2}}{\left(1+\frac{1}{x^2}\right)^{3/2}}$$

11. (15 points) Determine each limit or explain why the limit does not exist.

(a) $\displaystyle\lim_{(x,y)\to(2,1)} \frac{x-y-1}{\sqrt{x-y}-1}$ $\quad \frac{0}{0}$ MORE WORK

$$= \lim_{(x,y)\to(2,1)} \frac{x-y-1}{\sqrt{x-y}-1} \cdot \frac{\sqrt{x-y}+1}{\sqrt{x-y}+1} = \lim_{(x,y)\to(2,1)} \frac{\cancel{(x-y-1)}(\sqrt{x-y}+1)}{\cancel{(x-y-1)}}$$

$$= \sqrt{2-1}+1 = \boxed{2}$$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$ $\quad \frac{0}{0}$ MORE WORK

CONVERT TO POLAR

$$\lim_{r\to 0} \frac{r^3\cos\theta\sin^2\theta}{r^2} = \lim_{r\to 0} r\cos\theta\sin^2\theta = \boxed{0}$$

(c) $\displaystyle\lim_{(x,y)\to(0,1)} \frac{\sin(x^2+y-1)}{3y+x^2-3}$ $\quad \frac{0}{0}$ MORE WORK

ALONG $x=0$: $\displaystyle\lim_{y\to 1} \frac{\sin(y-1)}{3(y-1)} = \frac{1}{3}$

ALONG $y=1$: $\displaystyle\lim_{x\to 0} \frac{\sin x^2}{x^2} = 1$

LIMIT DNE

Yes

12. (2 points) True or false: The curvature at any point on a circle is constant and equal to the radius of the circle.

No

FALSE, $\kappa = \frac{1}{r}$ AND IS CONSTANT

13. (10 points) Let $g(x, y, z) = 4x^2 - y^2 + 4z^2$.

(a) What is the domain of g?

$\boxed{\mathbb{R}^3}$

(b) Evaluate $g(1, -3, 1)$.

$$g(1,-3,1) = 4(1)^2 - (-3)^2 + 4(1)^2$$
$$= 4 - 9 + 4 = \boxed{-1}$$

(c) Describe the level surface $g(x, y, z) = 9$.

$4x^2 - y^2 + 4z^2 = 9$ DESCRIBES A HYPERBOLOID OF ONE SHEET

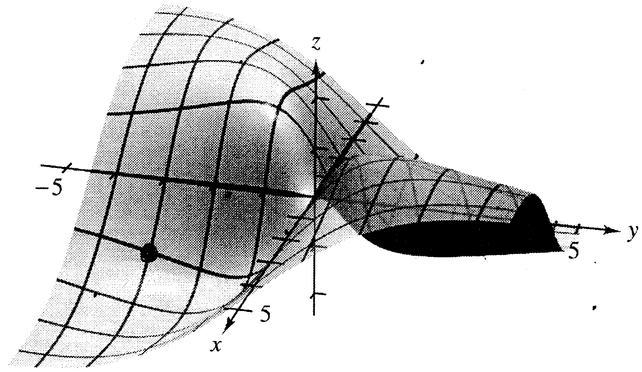
(d) Describe the level surface $g(x, y, z) = -9$.

$4x^2 - y^2 + 4z^2 = -9$ DESCRIBES A HYPERBOLOID OF TWO SHEETS

(e) At which points is g continuous?

g IS CONTINUOUS EVERYWHERE ON $\mathbb{R}^3$

14. (6 points) The graph of $z = f(x, y)$ is shown below. Look at the point on the graph where $x = 1$ and $y = -3$.



(a) Based on the graph, is the number $f_x(1, -3)$ positive or negative?

SLOPES DOWNWARD IN POS X DIRECTION

(b) Based on the graph, is the number $f_y(1, -3)$ positive or negative?

SLOPES DOWNWARD IN POS y DIRECTION

(c) Which is greater: $|f_x(1, -3)|$ or $|f_y(1, -3)|$?

STEEPER IN DIRECTION PARALLEL TO X-AXIS

15. (10 points) Let $h(x, y) = x \cos xy^2$.

(a) Compute h_x.

$$h_x(x,y) = \boxed{\cos xy^2 - xy^2 \sin xy^2}$$

(b) Compute h_y.

$$h_y(x,y) = x(-\sin xy^2)(2yx) = \boxed{-2x^2y \sin xy^2}$$

(c) Would you expect that $h_{xy} = h_{yx}$? Explain.

YES. BASED ON THE FORM OF h, I EXPECT h_{xy} & h_{yx} TO BE CONTINUOUS ON $\mathbb{R}^2$. BY OUR THEOREM, $h_{xy} = h_{yx}$.

(d) Compute h_{yx}.

$$h_{yx}(x,y) = -4xy \sin xy^2 + (-2x^2y)(\cos xy^2)(y^2)$$

$$= \boxed{-4xy \sin xy^2 - 2x^2y^3 \cos xy^2}$$