Math 216 - Quiz 3

September 15, 2010

Name key

Score _____

Show all work to receive full credit. Supply explanations when necessary.

1. (5 points) A 600-gallon tank is filled with 200 gallons of pure water. A spigot is opened above the tank, and a salt water solution containing 1.5 lb of salt per gallon begins flowing into the tank at a rate of 5 gal/min. Simultaneously, a drain is opened at the bottom of the tank allowing the solution to leave the tank at a rate of 2 gal/min. What will be the concentration of salt in the solution at the precise moment when tank reaches its maximum capacity?

$A(t)$ = AMOUNT OF SALT AT t

$V(t)$ = VOLUME AT TIME t

$= 200 + 3t$

TANK IS FULL WHEN

$600 = 200 + 3t$

$\Rightarrow t = \frac{400}{3}$ MIN

$$\frac{1.5\text{ lb}}{\text{gal}} \cdot \frac{5\text{ gal}}{\text{min}} = \frac{7.5\text{ lb}}{\text{min}}$$

$$\longrightarrow \boxed{\begin{array}{c} A(t) \\ A(0) = 0 \\ V(0) = 200 \end{array}}$$

$$\frac{2\text{ gal}}{\text{min}} \cdot \frac{A(t)\text{ lb}}{V(t)\text{ gal}}$$

$$\frac{2A}{200+3t} \ \frac{\text{lb}}{\text{min}}$$

$$\frac{dA}{dt} = 7.5 - \frac{2A}{200+3t}, \quad A(0) = 0$$

$$\frac{dA}{dt} + \frac{2}{200+3t} A = 7.5$$

GOTTA BE POSITIVE

$$\mu(t) = e^{\int \frac{2}{200+3t}\,dt} = e^{\frac{2}{3}\ln|200+3t|} = (200+3t)^{2/3}$$

$$(200+3t)^{2/3} A = \int 7.5(200+3t)^{2/3}\,dt$$

$$= \frac{7.5}{3} \cdot \frac{3}{5}(200+3t)^{5/3} + C$$

$$A(t) = 1.5(200+3t) + C(200+3t)^{-2/3}$$

$$A(0) = 0 \Rightarrow (1.5)(200) + C(200)^{-2/3} = 0 \Rightarrow C = -300(200)^{2/3}$$

$$= -10259.86$$

$$A(t) = 1.5(200+3t) - \frac{300(200)^{2/3}}{(200+3t)^{2/3}}$$

$$A\left(\frac{400}{3}\right) = 755.775 \text{ lbs}$$

CONCENTRATION IS THAT $\div 600$

$$\approx \boxed{1.26 \text{ lbs/gal}}$$

2. (3 points) Show that the following equation is exact. Then find the solution that satisfies the given condition.

$$(1 + e^x y + xe^x y)\,dx + (xe^x + 2)\,dy = 0, \quad y(0) = 1$$

$$\frac{\partial}{\partial y}\left(1 + e^x y + xe^x y\right) = e^x + xe^x = \frac{\partial}{\partial x}\left(xe^x + 2\right) \Rightarrow$$ EQUATION IS EXACT.

$$\frac{\partial F}{\partial y} = xe^x + 2 \Rightarrow \underbrace{F(x,y) = xe^x y + 2y + g(x)}$$

$$\frac{\partial F}{\partial x} = xe^x y + e^x y + g'(x)$$ AND THIS MUST EQUAL $1 + e^x y + xe^x y$

So $g'(x) = 1$ AND $g(x) = x$

$$F(x,y) = xe^x y + 2y + x = C$$

$y(0) = 1 \Rightarrow C = 2$

$$xe^x y + 2y + x = 2$$

$$\boxed{y = \frac{2 - x}{xe^x + 2}}$$

3. (2 points) Find the orthogonal trajectories of $y = \dfrac{Cx}{1+x}$.

$$(1+x)y = Cx$$

$$y + xy = Cx$$

$$\frac{dy}{dx} + x\frac{dy}{dx} + y = C$$

$$\frac{dy}{dx}(1+x) + y = \frac{(1+x)y}{x}$$

$$\frac{dy}{dx}(1+x) = \frac{(1+x)y}{x} - y = \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{y}{x(1+x)} \Rightarrow$$ ORTHO TRAJ'S SATISFY

$$\frac{dy}{dx} = -\frac{x(1+x)}{y}$$

$$y\,dy = (-x - x^2)\,dx$$

$$\boxed{\frac{y^2}{2} = -\frac{x^2}{2} - \frac{x^3}{3} + C}$$