Math 216 - Quiz 7

November 10, 2010

Name Key

Score _____

Show all work to receive full credit. Supply explanations when necessary.

1. (5 points) Solve the following initial value problem:

$$x^2y'' - (x^2 + 2x)y' + (x + 2)y = 0; \quad y(1) = 1,\ y'(1) = 2$$

(Hint: Find one solution by guessing and checking.)

IT IS EASY TO SEE THAT $y_1 = x$ IS A SOLUTION:

$$y_1 = x \Rightarrow y_1' = 1 \Rightarrow y_1'' = 0$$

$$x^2(0) - (x^2+2x)(1) + (x+2)(x) = 0 \checkmark$$

2ND SOLUTION:

STANDARD FORM $y'' + \left(-1 - \frac{2}{x}\right)y' + \left(\frac{1}{x} + \frac{2}{x^2}\right)y = 0$

$$p(x) = \left(-1 - \frac{2}{x}\right) \Rightarrow -\int p(x)\,dx = \int\left(1 + \frac{2}{x}\right)dx = x + 2\ln|x|$$

$$= x + \ln x^2$$

$$e^{-\int p(x)dx} = e^x e^{\ln x^2} = x^2 e^x$$

$$v(x) = \int \frac{1}{x^2} x^2 e^x\,dx = \int e^x\,dx = e^x$$

$$y_2 = xe^x$$

$$y(x) = c_1 x + c_2 x e^x$$

$$y(1) = 1 \Rightarrow 1 = c_1 + c_2 e$$

$$y'(x) = c_1 + c_2 e^x + c_2 x e^x$$

$$y'(1) = 2 \Rightarrow 2 = c_1 + 2c_2 e$$

$$2 = c_1 + 2c_2 e$$

$$1 = c_1 + c_2 e$$

$$1 = c_2 e \Rightarrow c_2 = \frac{1}{e}$$

$$\Rightarrow c_1 = 0$$

$$\boxed{y(x) = \frac{1}{e} x e^x = x e^{x-1}}$$

2. (5 points) Use variation of parameters to solve the following equation:

$$y'' + 2y' + y = e^{-x} \ln x$$

i) Homo Eq: $y'' + 2y' + y = 0$

$r^2 + 2r + 1 = 0 \quad (r+1)^2 = 0$

$r = -1, r = -1$

$$\underline{y_h(x) = c_1 e^{-x} + c_2 x e^{-x}}$$

ii) Non Homo / Variation of Param

$$g(x) = e^{-x} \ln x$$

$$y_1 = e^{-x} \qquad y_2 = x e^{-x}$$

$$W = \begin{bmatrix} e^{-x} & x e^{-x} \\ -e^{-x} & e^{-x} - x e^{-x} \end{bmatrix} = e^{-2x}$$

$$V_1(x) = \int \frac{-e^{-x} \ln x \; x e^{-x}}{e^{-2x}} dx$$

$$= \int -x \ln x \, dx = -\frac{x^2}{2} \ln x + \int \frac{x}{2} dx$$

$u = \ln x \quad du = \frac{1}{x} dx$

$dv = -x \quad v = -\frac{x^2}{2}$

$$V_1(x) = \frac{x^2}{4} - \frac{x^2}{2} \ln x$$

$$V_2(x) = \int \frac{e^{-x} \ln x \; e^{-x}}{e^{-2x}} dx = \int \ln x \, dx$$

$$= x \ln x - x$$

$$y_p(x) = \left(\frac{x^2}{4} - \frac{x^2}{2} \ln x\right) e^{-x} + (x \ln x - x) x e^{-x}$$

$$y_p(x) = -\frac{3}{4} x^2 e^{-x} + \frac{1}{2} x^2 \ln x \, e^{-x}$$

$$\boxed{y(x) = c_1 e^{-x} + c_2 x e^{-x} - \frac{3}{4} x^2 e^{-x} + \frac{1}{2} x^2 \ln x \, e^{-x}}$$