Math 216 - Quiz 8

November 17, 2010

Name key

Score _____

Show all work to receive full credit. Supply explanations when necessary.

1. (5 points) A 1-kg mass is attached to a spring with stiffness 10 N/m. The damping constant for the system is 2 N-sec/m. The mass is moved 2 m to the left of equilibrium (compressing the spring) and released from rest. Find the equation of motion of the mass. Write your final result in terms of a single trigonometric function with a phase shift. When does the mass pass through equilibrium for the first time?

$$x'' + 2x' + 10x = 0; \quad x(0) = -2, \; x'(0) = 0$$

$$r^2 + 2r + 10 = 0 \Rightarrow (r+1)^2 = -9$$

$$r + 1 = \pm 3i$$

$$r = -1 \pm 3i$$

$$\alpha = -1, \; \beta = 3$$

$$x(t) = c_1 e^{-t} \cos 3t + c_2 e^{-t} \sin 3t$$

$$x'(t) = -c_1 e^{-t} \cos 3t - 3c_1 e^{-t} \sin 3t - c_2 e^{-t} \sin 3t + 3c_2 e^{-t} \cos 3t$$

$$x(0) = -2 \Rightarrow c_1 = -2 = A \sin \varphi$$

$$x'(0) = 0 \Rightarrow -c_1 + 3c_2 = 0 \Rightarrow c_2 = -\frac{2}{3} = A \cos \varphi$$

$\sin \varphi, \cos \varphi < 0$

$\Rightarrow \varphi$ IN QUAD 3

$$A = \sqrt{(-2)^2 + \left(-\frac{2}{3}\right)^2} = \sqrt{4 + \frac{4}{9}} = \sqrt{\frac{40}{9}} = \frac{2}{3}\sqrt{10}$$

$\tan \varphi = 3$ & φ IN QUAD 3 $\Rightarrow \varphi = \pi + \tan^{-1} 3 \approx 4.3906$

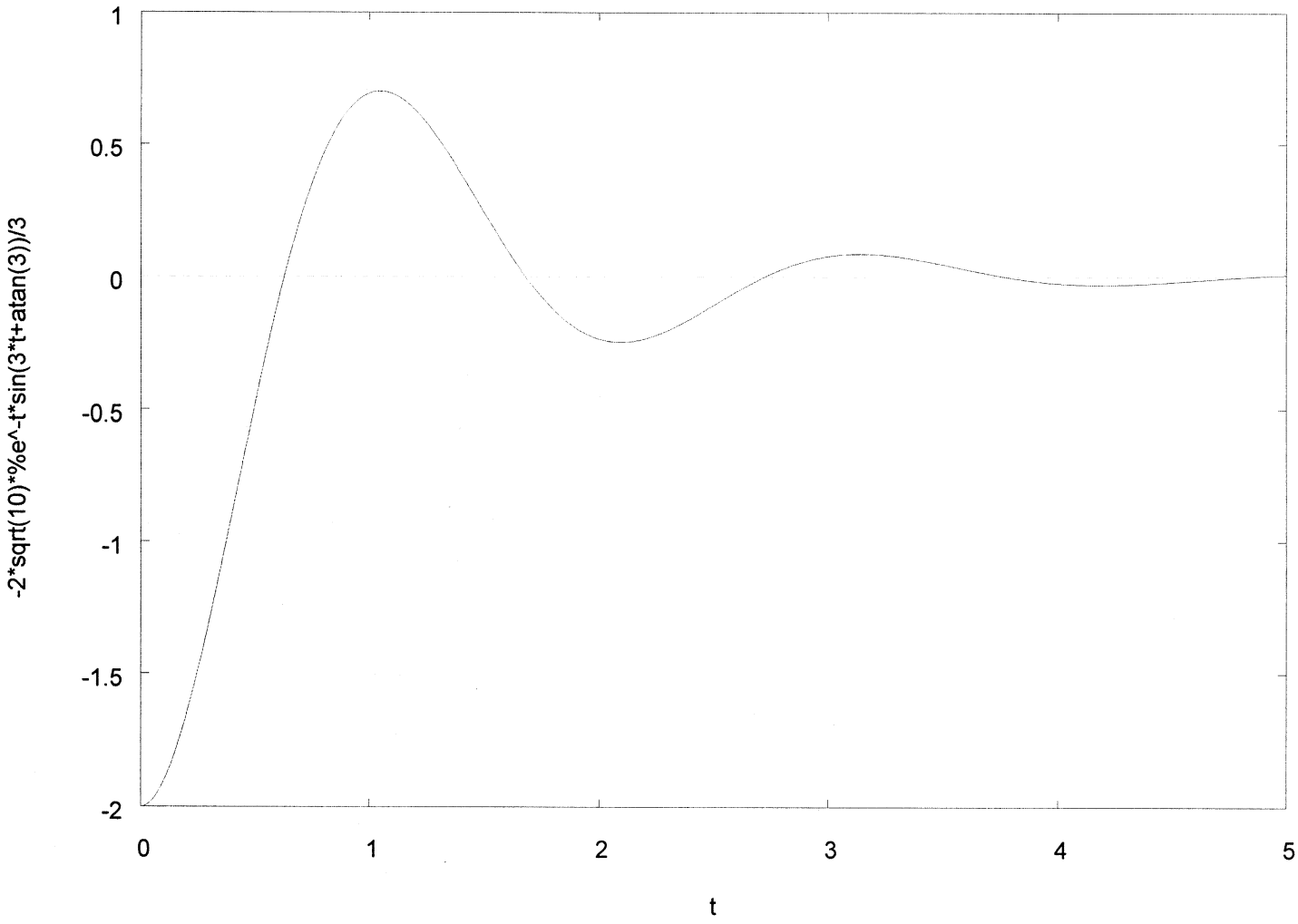
$$\boxed{x(t) = \frac{2\sqrt{10}}{3} e^{-t} \sin(3t + 4.3906)}$$

(GRAPH ATTACHED.)

$$x(t) = 0 \Rightarrow 3t + 4.3906 = k\pi$$

$$t = \frac{k\pi - 4.3906}{3}$$

$$k = 2 \Rightarrow \boxed{t \approx 0.63}$$


1
0.5
0
-0.5
-1
-1.5
-2
0
1
2
3
4
5
t
-2*sqrt(10)*%e^-t*sin(3*t+atan(3))/3

2. (5 points) The oscillations of a mass on a spring are described by the initial value problem

$$0.2\frac{d^2x}{dt^2} + 1.2\frac{dx}{dt} + 2x = 5\cos 4t, \qquad x(0) = 0.5, \quad x'(0) = 0.$$

Find the equation of motion, and identify the transient and steady-state terms.

$$x'' + 6x' + 10x = 25\cos 4t; \quad x(0) = 0.5, \; x'(0) = 0$$

Homo eq:

$$r^2 + 6r + 10 = 0 \Rightarrow (r+3)^2 = -1 \Rightarrow r = -3 \pm i \Rightarrow \alpha = -3, \; \beta = 1$$

$$\underline{x_h(t) = c_1 e^{-3t}\cos t + c_2 e^{-3t}\sin t}$$

NonHomo eq:

$$g(x) = 25\cos 4t \Rightarrow x_p(t) = A\sin 4t + B\cos 4t$$

$$x_p'(t) = 4A\cos 4t - 4B\sin 4t \qquad x_p''(t) = -16A\sin 4t - 16B\cos 4t$$

$$-16A\sin 4t - 16B\cos 4t + 24A\cos 4t - 24B\sin 4t + 10A\sin 4t + 10B\cos 4t = 25\cos 4t$$

$\sin 4t$: $-16A - 24B + 10A = 0$

$\cos 4t$: $-16B + 24A + 10B = 25$

$$\Rightarrow \begin{aligned} 6A + 24B &= 0 \\ 24A - 6B &= 25 \end{aligned} \Rightarrow \begin{aligned} 24A + 96B &= 0 \\ 24A - 6B &= 25 \end{aligned}$$

$$102B = -25$$

$$B = \frac{-25}{102} \Rightarrow A = -4B \Rightarrow A = \frac{100}{102}$$

$$\underline{x_p(t) = \frac{100}{102}\sin 4t - \frac{25}{102}\cos 4t}$$

$$x(t) = c_1 e^{-3t}\cos t + c_2 e^{-3t}\sin t + \frac{100}{102}\sin 4t - \frac{25}{102}\cos 4t$$

$$x(0) = 0.5 \Rightarrow c_1 - \frac{25}{102} = 0.5 \Rightarrow c_1 = \frac{38}{51}$$

Only writing nonzero terms...

$$x'(0) = 0 \Rightarrow -3c_1 + c_2 + \frac{400}{102} = 0 \Rightarrow c_2 = \frac{3(38)}{51} - \frac{400}{102} = -\frac{86}{51}$$

$$\boxed{x(t) = \underbrace{\frac{38}{51}e^{-3t}\cos t - \frac{86}{51}e^{-3t}\sin t}_{\text{TRANSIENT}} + \underbrace{\frac{100}{102}\sin 4t - \frac{25}{102}\cos 4t}_{\text{STEADY STATE}}}$$