Math 216 - Quiz 5

March 12, 2014

Name key

Score

Show all work to receive full credit. Supply explanations when necessary.

1. (4 points) A small cannonball weighing 16 pounds ($mg = 16$) is shot vertically upward from the ground with an initial velocity of 300 ft/s. Air resistance is proportional to the velocity with constant of proportionality $b = 0.0025$. Find a function giving the height of the cannonball at time t. Also determine the maximum height of the cannonball.

$$mg = 16 \Rightarrow m = \frac{1}{2}$$

$$\frac{1}{2}\frac{dv}{dt} = -16 - 0.0025v, \quad v(0) = 300$$

$$\frac{dv}{dt} + 0.005v = -32$$

$$\mu(t) = e^{\int 0.005\,dt} = e^{0.005t}$$

$$v(t) = e^{-0.005t}\int -32e^{0.005t}\,dt$$

$$= e^{-0.005t}\left(\frac{-32}{0.005}e^{0.005t} + C\right)$$

$$= -6400 + Ce^{-0.005t}$$

$$v(0) = 300 \Rightarrow C = 6700$$

$$v(t) = -6400 + 6700e^{-0.005t}$$

$$x(t) = -6400t - \frac{6700}{0.005}e^{-0.005t} + C$$

$$x(0) = 0 \Rightarrow C = \frac{6700}{0.005} = 1{,}340{,}000$$

$$\boxed{x(t) = -6400t - 1340000e^{-0.005t} + 1340000}$$

MAX HEIGHT OCCURS WHEN $v(t) = 0$

$$e^{-0.005t} = \frac{64}{67}$$

$$t = \frac{\ln(64/67)}{-0.005}$$

$$\approx 9.16$$

$$\boxed{x(9.16\ldots) \approx 1363.8 \text{ FT}}$$

2. (6 points) Consider the initial value problem $y' = x^2 + y^2$, $y(0) = 1$.

(a) Use the improved Euler's method with $h = 0.1$ to approximate $y(0.3)$.

$$y^* = y_N + h\, f(x_N, y_N)$$
$$x_{N+1} = x_N + h$$
$$y_{N+1} = y_N + \frac{h}{2}\left[f(x_N, y_N) + f(x_{N+1}, y^*)\right]$$
$$y_0 = 1$$
$$x_0 = 0$$

$$y^* = 1 + 0.1(1) = 1.1$$

$$x_1 = 0.1$$
$$y_1 = 1 + 0.1/2\left[(1) + (1.22)\right] = 1.111$$

$$y^* = 1.111 + 0.1(1.244321) = 1.2354321$$
$$x_2 = 0.2$$
$$y_2 = y_1 + 0.1/2\left[x_1^2 + y_1^2 + x_2^2 + y^{*2}\right] = 1.251530674$$

$$y^* = y_2 + 0.1(x_2^2 + y_2^2) = 1.412163576$$
$$x_3 = 0.3,\quad y_3 = y_2 + 0.1/2\left[x_2^2 + y_2^2 + x_3^2 + y^{*2}\right]$$

$$y(0.3) \approx y_3 = 1.436057423$$

$$\boxed{y(0.3) \approx 1.436}$$

(b) Use a computer program to experiment with the improved Euler's method with various step sizes. Try to find the solution of the equation $y(x) = 1.75$ accurate to 3 decimal places.

Using $h = 0.1$, A SOLUTION IS BETWEEN $x = 0.4$ & $x = 0.5$

Using $h = 0.01$, A SOLUTION IS BETWEEN $x = 0.41$ & $x = 0.42$

Using $h = 0.001$, A SOLUTION IS BETWEEN $x = 0.417$ & $x = 0.418$

Using $h = 0.0001$, A SOLUTION IS BETWEEN $x = 0.4171$ & $x = 0.4172$

$$\boxed{y(x) = 1.75 \;\Downarrow\; x \approx 0.417}$$

(c) Determine the Taylor method of order 3 for this differential equation.

$$f(x,y) = x^2 + y^2$$
$$f'(x,y) = 2x + 2y\frac{dy}{dx}$$
$$= 2x + 2x^2y + 2y^3$$
$$f''(x,y) = 2 + 4xy + 2x^2\frac{dy}{dx} + 6y^2\frac{dy}{dx}$$
$$= 2 + 4xy + 2x^4 + 2x^2y^2 + 6x^2y^2 + 6y^4$$
$$= 2 + 4xy + 2x^4 + 8x^2y^2 + 6y^4$$

$$\boxed{\begin{aligned} y_{N+1} &= y_N + h(x_N^2 + y_N^2) + \frac{h^2}{2}\left(2x_N + 2x_N^2 y_N + 2y_N^3\right) \\ &\quad + \frac{h^3}{6}\left(2 + 4x_N y_N + 2x_N^4 + 8x_N^2 y_N^2 + 6y_N^4\right) \\ x_{N+1} &= x_N + h \end{aligned}}$$