Math 216 - Test 1

February 19, 2014

Name key

Score

Show all work to receive full credit. Supply explanations where necessary. Give explicit solutions when possible. All integration must be done by hand, unless otherwise specified.

1. (10 points) State whether each equation is ordinary or partial, linear or nonlinear, and give its order.

   (a) $t^2w'' + 4tw' - 3w = t\cos 2t$

   ORDINARY, LINEAR, 2^{ND} ORDER

   (b) $\left(\dfrac{dy}{dx}\right)^2 - e^x y = 10x$

   ORDINARY, NONLINEAR, 1^{ST} ORDER

   (c) $vv'' + 5xv = x + 1$

   ORDINARY, NONLINEAR, 2^{ND} ORDER

   (d) $\dfrac{\partial^2 u}{\partial y^2} - \dfrac{\partial u}{\partial x} + x^2yu = 7y^2$

   PARTIAL, LINEAR, 2^{ND} ORDER

2. (3 points) Suppose you are sketching the direction field for the differential equation

$$x^2\frac{dy}{dx} + 3xy^3 = 4.$$

What is the slope of the solution curve passing through $(2, 3)$?

$$(2)^2\frac{dy}{dx} + 3(2)(3)^3 = 4$$

$$4\frac{dy}{dx} + 162 = 4$$

$$\frac{dy}{dx} = \frac{4-162}{4} = \boxed{-39.5}$$

3. (8 points) Analyze each initial value problem and determine whether we should expect a unique solution to exist through the given point.

(a) $\dfrac{dy}{dx} - xy = \sin^2 x, \quad y(\pi) = 3$

$$\left.\begin{aligned} f(x,y) &= \sin^2 x + xy \\ \frac{\partial f}{\partial y} &= x \end{aligned}\right\}$$

BOTH ARE CONTINUOUS EVERYWHERE, IN PARTICULAR AROUND $(\pi, 3)$.

THERE IS A UNIQUE SOLUTION THROUGH $(\pi, 3)$.

(b) $y\dfrac{dy}{dx} = x, \quad y(1) = 0$

$$\left.\begin{aligned} f(x,y) &= \frac{x}{y} \\ \frac{\partial f}{\partial y} &= -\frac{x}{y^2} \end{aligned}\right\}$$

NEITHER IS CONTINUOUS AT $(1,0)$. I DO NOT EXPECT A UNIQUE SOLUTION THROUGH $(1,0)$.

4. (12 points) Solve: $\dfrac{1}{x}\dfrac{dy}{dx} = \dfrac{y \sin x}{y^2+1}, \quad y(\pi) = 1$

SEPARABLE EQN

$$\frac{y^2+1}{y}\,dy = x \sin x \, dx$$

$$\int \left(y + \frac{1}{y}\right) dy = \int x \sin x \, dx = -x\cos x + \int \cos x \, dx$$

$u = x \qquad du = dx$

$dv = \sin x \, dx \qquad v = -\cos x$

$$\frac{y^2}{2} + \ln|y| = -x\cos x + \sin x + C$$

$$y(\pi) = 1 \Rightarrow \frac{1}{2} = \pi + C \Rightarrow C = \frac{1}{2} - \pi$$

$$\boxed{\frac{y^2}{2} + \ln|y| = \frac{1}{2} - \pi - x\cos x + \sin x}$$

5. (2 points) Is your solution above implicit or explicit?

$\boxed{\text{IMPLICIT}}$

6. (9 points) Use Euler's method with $h = 0.2$ to approximate the value of $y(1.4)$ for the initial value problem $y' = \frac{1}{x}(y^2 + y)$, $y(1) = 1$.

$y_0 = 1$

$X_0 = 1$

$y_1 = y_0 + (0.2)\left(\frac{1}{X_0}\right)\left(y_0^2 + y_0\right)$

$= 1 + 0.2(1)(2) = 1.4$

$X_1 = X_0 + 0.2 = 1.2$

$y_2 = y_1 + 0.2\left(\frac{1}{X_1}\right)\left(y_1^2 + y_1\right)$

$= 1.4 + 0.2\left(\frac{1}{1.2}\right)(3.36)$

$= 1.96 \longrightarrow \boxed{y(1.4) \approx 1.96}$

$X_2 = X_1 + 0.2 = 1.4$

7. (4 points) Without actually solving the IVP above, briefly explain how you would find the exact solution.

THE EQUATION IS SEPARABLE.

SOLVE BY SEPARATING VARIABLES.

$$\frac{1}{y^2 + y}\, dy = \frac{1}{X}\, dx$$

$$\int \frac{1}{y^2 + y}\, dy = \int \frac{1}{X}\, dx$$

PARTIAL FRACTIONS!

8. (16 points) Consider the following initial value problem:

$$(e^x y + xe^x y)\, dx + (xe^x + 2)\, dy = 0, \quad y(0) = 4.$$

(a) Use the test for exactness to show that the DE is exact?

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(e^x y + xe^x y\right) = e^x + xe^x = \frac{\partial}{\partial x}\left(xe^x + 2\right) = \frac{\partial N}{\partial x}$$

EXACT!

(b) Solve the initial value problem.

$$\frac{\partial F}{\partial y} = xe^x + 2 \Rightarrow F(x,y) = \underline{xe^x y + 2y + h(x)}$$

$$\frac{\partial F}{\partial x} = xe^x y + e^x y + h'(x) = M(x,y)$$

$$= xe^x y + e^x y$$

$$\Rightarrow h'(x) = 0$$

$$\Rightarrow h(x) = 0$$

$$F(x,y) = xe^x y + 2y$$

SOL'N IS $xe^x y + 2y = C$

$y(0) = 4 \Rightarrow 8 = C$

$$\boxed{xe^x y + 2y = 8}$$

(c) Is your solution explicit or implicit?

$xe^x y + 2y = 8$ IS IMPLICIT,

BUT $y = \dfrac{8}{xe^x + 2}$ IS EXPLICIT.

(d) Show that the equation is also separable.

$$(xe^x + 2)\, dy = -y\left(e^x + xe^x\right) dx$$

$$-\frac{1}{y}\, dy = \frac{e^x + xe^x}{xe^x + 2}\, dx$$

VARIABLES ARE SEPARATED!

(e) Is the equation also linear?

$$(xe^x + 2)\frac{dy}{dx} + \left(e^x + xe^x\right) y = 0$$

YES, IT'S LINEAR.

9. (11 points) Solve: $\frac{1}{x}\frac{dy}{dx} - \frac{2y}{x^2} = x\cos x$

MULT BY X TO GET

$$\frac{dy}{dx} - \frac{2}{x}y = x^2 \cos x$$

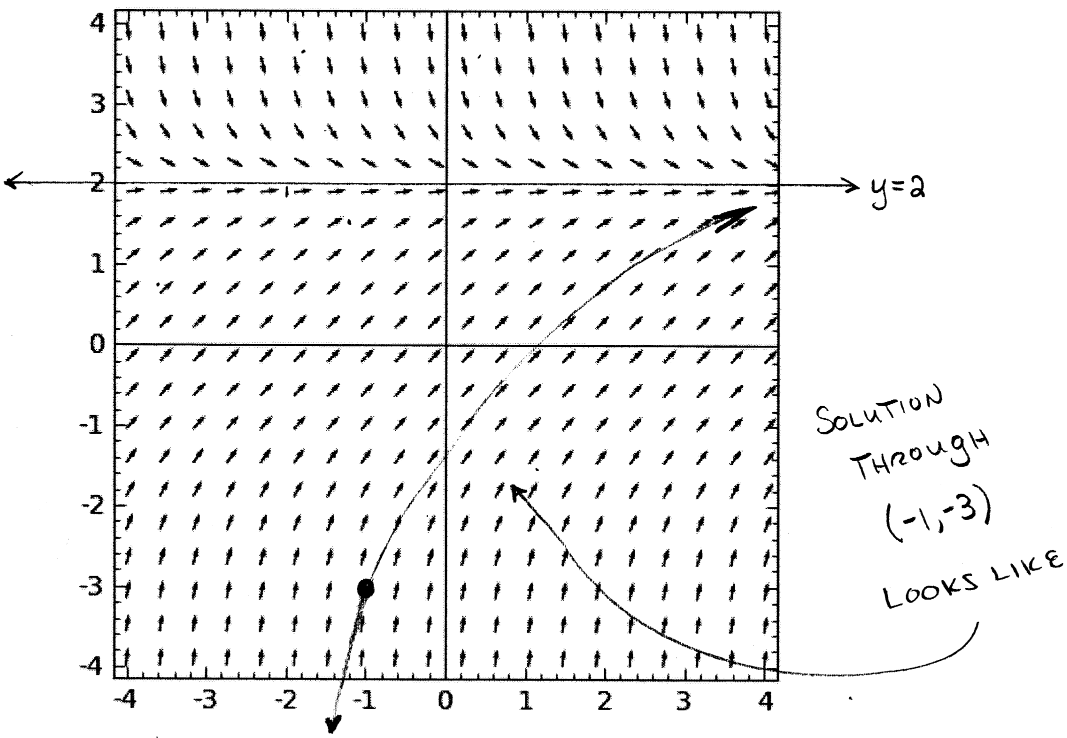
$$\mu(x) = e^{\int -\frac{2}{x}dx} = e^{-2\ln|x|} = \frac{1}{x^2}$$

$$y = x^2 \int \cos x \, dx$$

$$y = x^2(\sin x + C)$$

$$\boxed{y = x^2 \sin x + Cx^2}$$

10. (9 points) A direction field for $\dfrac{dy}{dx} = 1 - \dfrac{y^3}{8}$ is shown below. Use the direction field to solve the following problems.



(a) What is the unique solution passing through $(-2, 2)$?

THE CONSTANT FUNCTION $y = 2$

(b) Suppose you have found a solution, $y(x)$, passing through a given point. What can you say about $\lim_{x\to\infty} y(x)$?

IT LOOKS LIKE FOR ANY SOLUTION y,

$$\boxed{\lim_{x\to\infty} y(x) = 2}$$

(c) Sketch the solution curve passing through $(-1, -3)$.

SEE ABOVE.

11. (16 points) Consider the following initial value problem:

$$xy' - y = 3, \quad y(1) = -1.$$

$y' = \frac{3+y}{x}$

(a) Use Euler's method with $h = 1$ to approximate $y(3)$.

$y_0 = -1$

$x_0 = 1$

$y_1 = y_0 + h\left(\frac{3+y_0}{x_0}\right)$

$y_1 = -1 + (1)\left(\frac{3+(-1)}{1}\right)$

$y_1 = 1$

$x_1 = 2$

$y_2 = y_1 + h\left(\frac{3+y_1}{x_1}\right)$

$y_2 = 1 + 1\left(\frac{3+1}{2}\right) = 3$

$y_2 = 3$

$x_2 = 3$

$\boxed{y(3) \approx 3}$

(b) Use Euler's method with $h = 2$ to approximate $y(3)$.

$y_0 = -1$

$x_0 = 1$

$y_1 = y_0 + h\left(\frac{3+y_0}{x_0}\right)$

$y_1 = -1 + 2\left(\frac{3+(-1)}{1}\right)$

$y_1 = 3$

$x_1 = 3$

$\boxed{y(3) \approx 3}$

(c) By solving the equation, find the exact value of $y(3)$.

$\frac{1}{3+y}\,dy = \frac{1}{x}\,dx$

$\ln|3+y| = \ln|x| + C_1$

$|3+y| = C_2|x|$

$y+3 = C_3 x$

$y = C_3 x - 3$

$y(1) = -1 = C_3 - 3$

$\Rightarrow C_3 = 2$

$$\boxed{\begin{gathered} y = 2x - 3 \\ y(3) = 2(3) - 3 = 3 \end{gathered}}$$

(d) Explain why Euler's method worked so well even with a big stepsize.

THE EXACT SOLUTION IS A LINEAR FUNCTION.

EULER'S METHOD IS APPROXIMATING A LINE

WITH A LINE --- IT WORKS PERFECTLY!