Math 216 - Test 2a
March 19, 2014

Name: key

Score: ______

Show all work to receive full credit. Supply explanations where necessary.

1. (10 points) Solve: $\dfrac{dy}{dx} = \dfrac{3y^2 - x^2}{2xy}$

$$\frac{dy}{dx} = \frac{3y}{2x} - \frac{x}{2y}$$

$$u = \frac{y}{x} \Rightarrow ux = y$$

$$\Rightarrow u + x\frac{du}{dx} = \frac{dy}{dx}$$

$$u + x\frac{du}{dx} = \frac{3}{2}u - \frac{1}{2u}$$

$$x\frac{du}{dx} = \frac{u}{2} - \frac{1}{2u}$$

$$x\frac{du}{dx} = \frac{u^2-1}{2u}$$

$$\frac{2u}{u^2-1}\,du = \frac{1}{x}\,dx$$

$$\int \frac{2u}{u^2-1}\,du = \int \frac{1}{x}\,dx$$

$w = u^2 - 1$

$dw = 2u\,du$

$$\int \frac{1}{w}\,dw = \int \frac{1}{x}\,dx$$

$$\ln|w| = \ln|x| + C$$

$$w = Cx$$

$$u^2 - 1 = Cx$$

$$u^2 = 1 + Cx$$

$$\frac{y}{x} = \sqrt{1+Cx}\,,$$ Assuming $\frac{y}{x} > 0$

$$\boxed{y = x\sqrt{1+Cx}}$$

2. (12 points) Solve: $xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0, \quad y(1) = 1$

$$\frac{\partial M}{\partial y} = x \qquad \frac{\partial N}{\partial x} = 4x \qquad \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M} = \frac{-3x}{-xy} = \frac{3}{y}$$

$$\mu(y) = e^{\int \frac{3}{y}\,dy} = y^3 \quad , \; y > 0$$

MULT BY y^3...

$$xy^4\,dx + (2x^2y^3 + 3y^5 - 20y^3)\,dy = 0$$

$$\frac{\partial}{\partial y}(xy^4) = 4xy^3 = \frac{\partial}{\partial x}(2x^2y^3 + 3y^5 - 20y^3)$$

NEW EQUATION IS EXACT!

$$\frac{\partial F}{\partial x} = xy^4 \Rightarrow F(x,y) = \frac{1}{2}x^2y^4 + g(y)$$

$$\frac{\partial F}{\partial y} = 2x^2y^3 + 3y^5 - 20y^3 \Rightarrow F(x,y) = \frac{2}{4}x^2y^4 + \frac{3}{6}y^6 - \frac{20}{4}y^4 + h(x)$$

$$F(x,y) = \frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4$$

$$F(1,1) = \frac{1}{2} + \frac{1}{2} - 5 = -4$$

SOL'N IS

$$\frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 = -4$$

3. (8 points) Suppose n is a positive integer. Find the family of curves that are the orthogonal trajectories of $y = cx^n$.

$$\frac{dy}{dx} = NCX^{N-1}.$$

$$C = \frac{y}{X^N}. \Rightarrow \frac{dy}{dx} = \frac{Nyx^{N-1}}{X^N} = \frac{Ny}{X}$$

ORTHO TRAJS SATISFY

$$\frac{dy}{dx} = -\frac{X}{Ny}$$

$$Ny\,dy = -x\,dx$$

$$\frac{N}{2}y^2 = -\frac{1}{2}x^2 + C_1$$

$$Ny^2 = -x^2 + C_2$$

$$\boxed{x^2 + Ny^2 = C_2}$$

4. (10 points) Determine the recursive formulas for the Taylor method of order 3 for the initial value problem

$$y' = x^2 + y, \quad y(0) = 0.$$

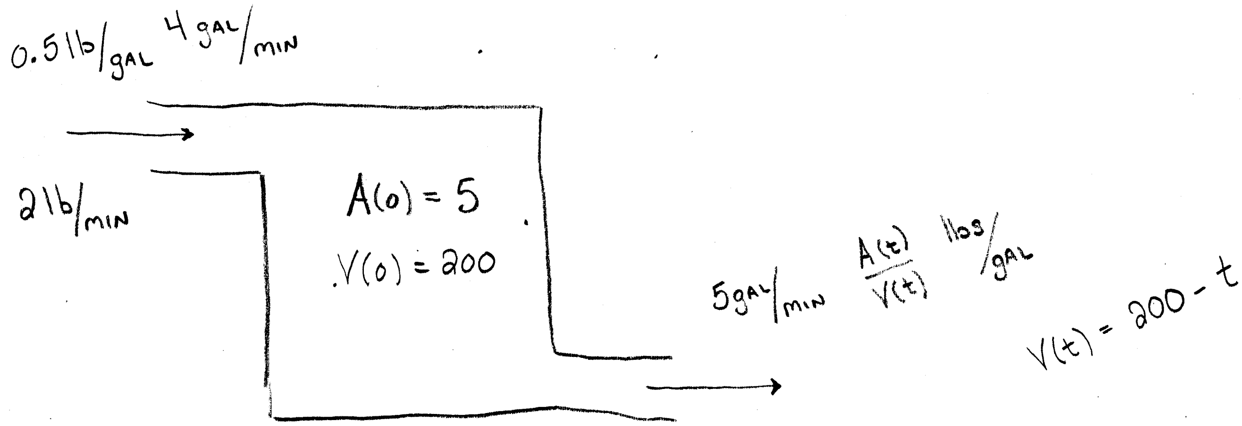
$$f(x,y) = x^2 + y$$

$$f'(x,y) = 2x + \frac{dy}{dx} = x^2 + 2x + y$$

$$f''(x,y) = 2x + 2 + \frac{dy}{dx} = x^2 + 2x + 2 + y$$

$$\boxed{\begin{aligned} y_{N+1} &= y_N + h\,(x_N^2 + y_N) + \frac{h^2}{2}\,(x_N^2 + 2x_N + y_N) \\ &\quad + \frac{h^3}{6}\,(x_N^2 + 2x_N + 2 + y_N) \\ x_{N+1} &= h + x_N \end{aligned}}$$

5. (15 points) A tank initially contains 5 lb of salt dissolved in 200 gal of water. A brine solution containing 0.5 lb of salt per gallon enters the tank at a rate of 4 gal/min. The well-stirred solution leaves the tank at 5 gal/min. Find the concentration of salt in the tank after 100 minutes.



$$\frac{dA}{dt} = 2 - \frac{5A}{200-t}, \quad A(0) = 5$$

$$A' + \frac{5}{200-t} A = 2$$

$$\mu(t) = e^{\int \frac{5}{200-t} dt} = e^{-5 \ln(200-t)} = \frac{1}{(200-t)^5}, \quad 0 \leq t \leq 200$$

$$A = (200-t)^5 \int 2(200-t)^{-5} dt$$

$$= (200-t)^5 \left[\frac{1}{2}(200-t)^{-4} + C \right]$$

$$= \frac{1}{2}(200-t) + (200-t)^5 (c)$$

$$A(0) = 5 \Rightarrow 100 + 200^5 (c) = 5$$

$$C = \frac{-95}{200^5}$$

$$\boxed{A(t) = \frac{1}{2}(200-t) - \frac{95(200-t)^5}{200^5}}$$

$$A(100) = 47.03125$$

$$\boxed{\frac{A(100)}{V(100)} = 0.4703125 \text{ lbs/GAL}}$$

Math 216 - Test 2b

March 19, 2014

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (12 points) Solve: $xy^2y' + y^3 = x\cos x$

$$y^2 y' + \frac{1}{x} y^3 = \cos x$$

$$3y^2 y' + \frac{3}{x} y^3 = 3\cos x$$

$$u = y^3$$

$$\frac{du}{dx} = 3y^2 \frac{dy}{dx}$$

$$\frac{du}{dx} + \frac{3}{x} u = 3\cos x$$

$$\mu(x) = e^{\int \frac{3}{x}dx} = e^{3\ln|x|} = x^3, \quad x > 0$$

$$u(x) = \frac{1}{x^3}\int 3x^3 \cos x \, dx = \frac{1}{x^3}\left(3x^3 \sin x + 9x^2 \cos x - 18x \sin x - 18\cos x + C\right)$$

+	$3x^3$	$\cos x$
−	$9x^2$	$\sin x$
+	$18x$	$-\cos x$
−	18	$-\sin x$
+	0	$\cos x$

$$u(x) = 3\sin x + \frac{9}{x}\cos x - \frac{18}{x^2}\sin x - \frac{18}{x^3}\cos x + \frac{C}{x^3}$$

$$y(x) = \sqrt[3]{3\sin x + \frac{9}{x}\cos x - \frac{18}{x^2}\sin x - \frac{18}{x^3}\cos x + \frac{C}{x^3}}$$

2. (12 points) Consider the one-parameter family of parabolas described by

$$y^2 = 4c(x + c).$$

Find the family of orthogonal trajectories. (Hint: The original family of curves can be rewritten $\sqrt{x^2 + y^2} - x = C$.)

$$y^2 = 4cx + 4c^2$$

$$y^2 + x^2 = x^2 + 4cx + 4c^2 = (x + 2c)^2$$

$$\sqrt{y^2 + x^2} = \pm x + C$$

$$\sqrt{y^2 + x^2} \mp x = C$$

$$\frac{1}{2}(y^2 + x^2)^{-1/2}\left(2y\frac{dy}{dx} + 2x\right) \mp 1 = 0$$

SOLVE FOR $\frac{dy}{dx}$ TO GET

$$\frac{dy}{dx} = \frac{-x \pm \sqrt{y^2 + x^2}}{y}.$$

ORTHO TRAJECTORIES SATISFY

$$\frac{dy}{dx} = \frac{y}{x \mp \sqrt{y^2 + x^2}} \cdot \frac{x \pm \sqrt{y^2 + x^2}}{x \pm \sqrt{y^2 + x^2}}$$

$$= \frac{y\left(x \pm \sqrt{y^2 + x^2}\right)}{-y^2}$$

$$= \frac{-x \mp \sqrt{y^2 + x^2}}{y}$$

SO, THE DE FOR $\sqrt{y^2 + x^2} - x = C$

IS $$\frac{dy}{dx} = \frac{-x + \sqrt{y^2 + x^2}}{y}$$

AND THE ORTHO TRAJS SATISFY

$$\frac{dy}{dx} = \frac{-x - \sqrt{y^2 + x^2}}{y},$$ WHICH

IS THE DE FOR

$$\sqrt{y^2 + x^2} + x = C.$$

THIS STATEMENT IS ALSO TRUE WHEN READ IN REVERSE.

SO,

$\sqrt{y^2 + x^2} = \pm x + C$ IS THE FAMILY OF ORTHO TRAJS FOR $\sqrt{y^2 + x^2} = \mp x + C$.

THIS ORIGINAL FAMILY IS SELF-ORTHOGONAL!

3. (11 points) Find an integrating factor of the form $x^n y^m$ that makes the following equation exact:

$$(12 + 5xy)\,dx + (6xy^{-1} + 3x^2)\,dy = 0.$$

Solve the equation.

$$\left(12x^N y^m + 5x^{N+1} y^{m+1}\right) dx + \left(6x^{N+1} y^{m-1} + 3x^{N+2} y^m\right) dy = 0$$

$$\frac{\partial}{\partial y}\left(12x^N y^m + 5x^{N+1} y^{m+1}\right) = 12m\, x^N y^{m-1} + (m+1)5x^{N+1} y^m$$

$$\frac{\partial}{\partial x}\left(6x^{N+1} y^{m-1} + 3x^{N+2} y^m\right) = (N+1)6x^N y^{m-1} + 3(N+2)x^{N+1} y^m$$

THESE MUST BE EQUAL.

$$12m = 6(N+1)$$
$$5(m+1) = 3(N+2)$$

$$12m - 6N = 6$$
$$-2(5m - 3N = 1)$$

$$2m = 4 \Rightarrow m = 2$$
$$N = 3$$

$$\left(12x^3y^2 + 5x^4y^3\right) dx + \left(6x^4y + 3x^5y^2\right) dy = 0$$

$$\frac{\partial F}{\partial x} = 12x^3y^2 + 5x^4y^3 \Rightarrow F(x,y) = 3x^4y^2 + x^5y^3 + g(y)$$

$$\frac{\partial F}{\partial y} = 6x^4y + 3x^5y^2 \Rightarrow F(x,y) = 3x^4y^2 + x^5y^3 + h(x)$$

$$\boxed{\text{SOL'N IS } \quad 3x^4y^2 + x^5y^3 = C}$$

4. (5 points) An object is launched upward so that its velocity (in m/s) at time t is described by the initial value problem

$$\frac{dv}{dt} = -19.6 - \frac{v}{50}, \quad v(0) = 200.$$

Experiment with a Runge-Kutta method of order 4 in order to find the time at which the object reaches its maximum height. Try to get 2 decimal digits of accuracy.

THE MAX HEIGHT WILL OCCUR WHEN $V(t) = 0$.

USING RK4 WITH $h = 0.001$ AND $N = 10000$,

WE FIND THAT $V(t) = 0$ WHEN

t IS BETWEEN 9.285 AND 9.286

$$\therefore \boxed{t \approx 9.29\ s}$$

5. (5 points) The following DE is homogeneous. Make the appropriate change of variables to obtain a separable equation. Then separate the variables and stop. You need not solve the equation.

$$\frac{2(x+2y)}{x-y} = \frac{dy}{dx}$$

$$\frac{\frac{1}{x}(2)(x+2y)}{\frac{1}{x}(x-y)} = \frac{2\left(1+2\frac{y}{x}\right)}{\left(1-\frac{y}{x}\right)} = \frac{dy}{dx}$$

$$u = \frac{y}{x} \qquad ux = y$$

$$u + x\frac{du}{dx} = \frac{dy}{dx}$$

$$\frac{2(1+2u)}{1-u} = u + x\frac{du}{dx}$$

$$\frac{2+4u}{1-u} - \frac{u-u^2}{1-u} = x\frac{du}{dx} \Rightarrow \frac{2+3u+u^2}{1-u} = x\frac{du}{dx} \Rightarrow \boxed{\frac{1}{x}dx = \frac{1-u}{2+3u+u^2}du}$$