Math 216 - Test 3

April 30, 2014

Name key

Score ______

Show all work. Supply explanations when necessary. You must work individually on this exam.

1. (10 points) Solve: $y'' - 2y' + y = 0;\ y(0) = 5,\ y'(0) = 10$

$r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$r = 1,\ r = 1$

$y(t) = c_1 e^t + c_2 t e^t$

$y(0) = 5 \Rightarrow c_1 = 5$

$y'(t) = c_1 e^t + c_2 t e^t + c_2 e^t$

$y'(0) = 10 \Rightarrow c_1 + c_2 = 10$

$\Rightarrow c_2 = 5$

$$\boxed{y(t) = 5e^t + 5te^t}$$

2. (14 points) Solve: $y'' - 5y' + 4y = 8e^x + 5x + 2$

Homo eq: $y'' - 5y' + 4y = 0$

$r^2 - 5r + 4 = 0$

$(r-4)(r-1) = 0$

$r = 4,\ r = 1$

$y_h(x) = c_1 e^{4x} + c_2 e^x$

Nonhomo eq: $g(x) = 8e^x + 5x + 2$

$y_p(x) = xAe^x + Bx + C$

$y_p'(x) = Ae^x + xAe^x + B$

$y_p''(x) = Ae^x + Ae^x + xAe^x$

$y_p''(x) - 5y_p'(x) + 4y_p(x) = 8e^x + 5x + 2$

$\Downarrow$

$2Ae^x + xAe^x - 5Ae^x - 5xAe^x - 5B$

$+ 4xAe^x + 4Bx + 4C = 8e^x + 5x + 2$

$\Downarrow$

$-3Ae^x + 4Bx + 4C - 5B = 8e^x + 5x + 2$

$-3A = 8 \qquad A = -\frac{8}{3}$

$4B = 5 \qquad B = \frac{5}{4}$

$4C - 5B = 2$

$4C = 2 + \frac{25}{4} \Rightarrow C = \frac{33}{16}$

$y(x) = y_h(x) + y_p(x)$

$$y(x) = c_1 e^{4x} + c_2 e^x - \frac{8}{3} x e^x + \frac{5}{4} x + \frac{33}{16}$$

3. (14 points) Solve: $y'' + 3y' + 2y = \sin e^x$

Homo eq: $y'' + 3y' + 2y = 0$

$r^2 + 3r + 2 = 0$

$(r+1)(r+2) = 0$

$r = -1,\ r = -2$

$y_h(x) = c_1 e^{-x} + c_2 e^{-2x}$

Nonhomo eq: $g(x) = \sin e^x$ VARIATION OF PARAMETERS...

$$W[e^{-x}, e^{-2x}](x) = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -2e^{-3x} + e^{-3x} = -e^{-3x}$$

$$V_1(x) = \int \frac{-e^{-2x}\sin e^x}{-e^{-3x}}\,dx = \int e^x \sin e^x\,dx = \int \sin u\,du = -\cos u = -\cos e^x$$

$u = e^x$
$du = e^x\,dx$

$$V_2(x) = \int \frac{e^{-x}\sin e^x}{-e^{-3x}}\,dx = \int -e^{2x}\sin e^x\,dx = -\int u \sin u\,du$$

INT-BY-PARTS

$u = e^x$
$du = e^x\,dx$

$$= -\left[-u\cos u + \sin u\right] = e^x \cos e^x - \sin e^x$$

$$y_p(x) = \cancel{-e^{-x}\cos e^x} + \cancel{e^{-2x}e^x\cos e^x} - e^{-2x}\sin e^x$$

$$\boxed{y(x) = c_1 e^{-x} + c_2 e^{-2x} - e^{-2x}\sin e^x}$$

4. (10 points) Solve the Cauchy-Euler equation:

$$4x^2y'' + 17y = 0;\ y(1) = -1,\ y'(1) = -\frac{1}{2}$$

$X = e^t$ TRANSFORMS THE EQUATION TO

$$4\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 17y = 0$$

$$4r^2 - 4r + 17 = 0 \Rightarrow r = \frac{4 \pm \sqrt{16 - 4(4)(17)}}{8} = \frac{4 \pm 16i}{8}$$

$$= \frac{1}{2} \pm 2i$$

$$y(t) = c_1 e^{\frac{1}{2}t} \cos 2t + c_2 e^{\frac{1}{2}t} \sin 2t$$

$$\Downarrow$$

$$y(x) = c_1\sqrt{x} \cos(\ln x^2) + c_2 \sqrt{x} \sin(\ln x^2)$$

$$y(1) = -1 = c_1 = -1$$

$$y'(x) = \frac{1}{2}c_1 x^{-1/2} \cos(\ln x^2) - \frac{c_1\sqrt{x}\, 2x}{x^2} \sin(\ln x^2) + \frac{1}{2}c_2 x^{-1/2} \sin(\ln x^2) + \frac{c_2\sqrt{x}\, 2x}{x^2} \cos(\ln x^2)$$

$$y'(1) = \frac{1}{2}c_1 + 2c_2 = -\frac{1}{2} \Rightarrow c_2 = 0$$

$$\boxed{y(x) = -\sqrt{x}\cos(\ln x^2)}$$

5. (12 points) The equation $y''' - 6y'' + 11y' - 6y = 3x$ has a particular solution $y_p(x) = -\frac{11}{12} - \frac{1}{2}x$. Find the general solution of the equation

$$y''' - 6y'' + 11y' - 6y = 12x.$$

Homo eq: $y''' - 6y'' + 11y' - 6y = 0$

$$\underbrace{r^3 - 6r^2 + 11r - 6 = 0}_{r=1 \text{ IS A SOLUTION}}$$

$$\begin{array}{r} r^2 - 5r + 6 \\ r-1 \overline{) r^3 - 6r^2 + 11r - 6} \\ \underline{r^3 - r^2} \\ -5r^2 + 11r - 6 \\ \underline{-5r^2 + 5r} \\ 6r - 6 \\ \underline{6r - 6} \\ 0 \end{array}$$

$(r-1)(r^2 - 5r + 6) = 0$

$(r-1)(r-2)(r-3) = 0$

$r = 1, r = 2, r = 3$

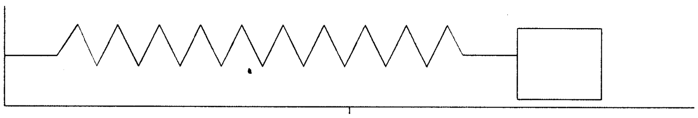
$y_h(x) = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$

NON HOMO: $g(x) = 12x$

$\Rightarrow \quad y_p(x) = 4\left(-\frac{11}{12} - \frac{1}{2}x\right) = -\frac{11}{3} - 2x$

$$\boxed{y(x) = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} - \frac{11}{3} - 2x}$$

6. (13 points) A 1/2-kg mass is attached to a spring with spring constant 5 N/m. The damping constant for the system is 1 N-sec/m. The mass is moved 2 m to the left of equilibrium (compressing the spring) and released from rest. Find the equation of motion. Write your final result in terms of a single trig function with phase shift. Graph your solution[1] and attach a copy.



$x = 0$
(Equilibrium)

$\frac{1}{2}x'' + x' + 5x = 0, \quad x(0) = -2, \quad x'(0) = 0$

$x'' + 2x' + 10x = 0$

$r^2 + 2r + 1 = -9$

$(r+1)^2 = -9$

$r + 1 = \pm 3i$

$r = -1 \pm 3i$

$\alpha = -1, \ \beta = 3$

$x(t) = c_1 e^{-t} \cos 3t + c_2 e^{-t} \sin 3t$

$x(0) = -2 \Rightarrow c_1 = -2$

$x'(t) = -c_1 e^{-t} \cos 3t - 3c_1 e^{-t} \sin 3t - c_2 e^{-t} \sin 3t + 3c_2 e^{-t} \cos 3t$

$x'(0) = 0 \Rightarrow -c_1 + 3c_2 = 0 \Rightarrow c_2 = -\frac{2}{3}$

$x(t) = -2e^{-t} \cos 3t - \frac{2}{3} e^{-t} \sin 3t$

$A = \sqrt{4 + \frac{4}{9}} = \sqrt{\frac{40}{9}} = \frac{2\sqrt{10}}{3}$

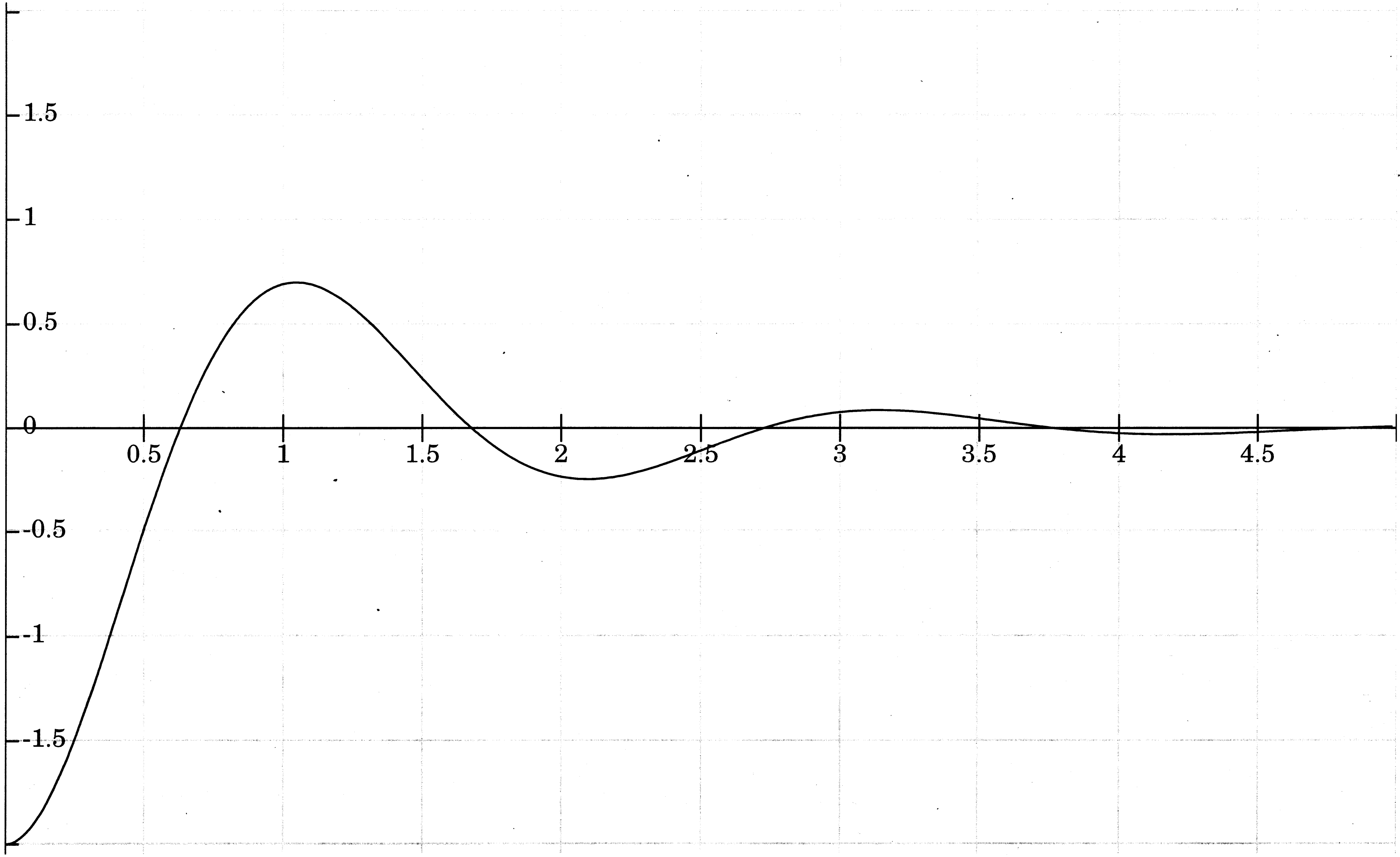
$\tan \varphi = \frac{2}{2/3} = 3$

BECAUSE C_1 & C_2 ARE BOTH NEG, φ IS A QUAD III ∠.

$\Rightarrow \varphi = (\pi + \tan^{-1} 3)$

$x(t) = \frac{2\sqrt{10}}{3} e^{-t} \sin\left(3t + \pi + \tan^{-1} 3\right)$

[1]If you don't have a good plotting program, try one that is available online such as http://fooplot.com.


1.5
1
0.5
0
-0.5
-1
-1.5
0.5
1
1.5
2
2.5
3
3.5
4
4.5

7. (12 points) Consider the equation $x^2y'' - 4xy' + 6y = 0,\ 0 < x < \infty$. Do not solve this differential equation.

(a) Verify that $y_1(x) = x^3$ and $y_2(x) = x^2$ are solutions.

$y_1 = x^3$
$y_1' = 3x^2$
$y_1'' = 6x$

$x^2y_1'' - 4xy_1' + 6y_1$
$= 6x^3 - 12x^3 + 6x^3$
$= 0$ ✓. y_1 IS A SOL'N.

$y_2 = x^2$
$y_2' = 2x$
$y_2'' = 2$

$x^2y_2'' - 4xy_2' + 6y_2$
$= 2x^2 - 8x^2 + 6x^2$
$= 0$ ✓

y_2 IS A SOL'N.

(b) Use the Wronskian to show that y_1 and y_2 are linearly independent on $(0, \infty)$.

$$W[y_1, y_2](x) = \begin{vmatrix} x^3 & x^2 \\ 3x^2 & 2x \end{vmatrix} = 2x^4 - 3x^4 = -x^4 \neq 0$$

ON $(0,\infty)$

$\therefore$ y_1 & y_2 ARE LIN. INDEP.

(c) Use what you've learned in parts (a) and (b) to find the solution of the IVP $x^2y'' - 4xy' + 6y = 0;\ y(1) = 5,\ y'(1) = 16$.

$y(x) = c_1x^3 + c_2x^2$

$y(1) = 5 \Rightarrow c_1 + c_2 = 5$

$y'(x) = 3c_1x^2 + 2c_2x$

$y'(1) = 16 \Rightarrow 3c_1 + 2c_2 = 16$

$c_1 + c_2 = 5$
$3c_1 + 2c_2 = 16$

$-c_1 = -6 \quad c_1 = 6 \Rightarrow c_2 = -1$

$$\boxed{y(x) = 6x^3 - x^2}$$

(d) Is your solution in part (c) unique? Explain.

IN STANDARD FORM, THE EQUATION IS

$$y'' - \frac{4}{x}y' + \frac{6}{x^2}y = 0.$$

THE COEFFICIENT FUNCTIONS ARE CONTINUOUS ON $(0,\infty)$.

$\therefore$ BY THEOREM 1 (SEC 6.1 p. 318), THE SOL'N IS UNIQUE.

8. (15 points) A forced mass-spring system is governed by the IVP

$$0.2\frac{d^2x}{dt^2} + 1.2\frac{dx}{dt} + 2x = 5\cos 4t;\ x(0) = \frac{1}{2},\ x'(0) = 0.$$

Find the transient and steady-state solutions. Identify which is which. Also sketch the graph of the equation of motion.

$$0.2x'' + 1.2x' + 2x = 5\cos 4t$$

$$x'' + 6x' + 10x = 25\cos 4t$$

Homo eq: $x'' + 6x' + 10x = 0$

$$r^2 + 6r + 9 = -1$$

$$(r+3)^2 = -1$$

$$r = -3 \pm i$$

$$x_h(t) = c_1 e^{-3t}\cos t + c_2 e^{-3t}\sin t$$

Non Homo eq: $g(t) = 5\cos 4t$

$$x_p(t) = A\cos 4t + B\sin 4t$$

$$x_p'(t) = -4A\sin 4t + 4B\cos 4t$$

$$x_p''(t) = -16A\cos 4t - 16B\sin 4t$$

$$x'' + 6x' + 10x = 25\cos 4t$$

$$\Downarrow$$

$$(-16A + 24B + 10A)\cos 4t + (-16B - 24A + 10B)\sin 4t = 25\cos 4t$$

$$-6A + 24B = 25$$

$$-24A - 6B = 0$$

$$102B = 100$$

$$B = \frac{100}{102} = \frac{50}{51}$$

$$\Rightarrow A = -\frac{25}{102}$$

$$x(t) = c_1 e^{-3t}\cos t + c_2 e^{-3t}\sin t - \frac{25}{102}\cos 4t + \frac{50}{51}\sin 4t$$

$$x(0) = \frac{1}{2} \Rightarrow c_1 - \frac{25}{102} = \frac{1}{2}$$

$$\Rightarrow c_1 = \frac{38}{51}$$

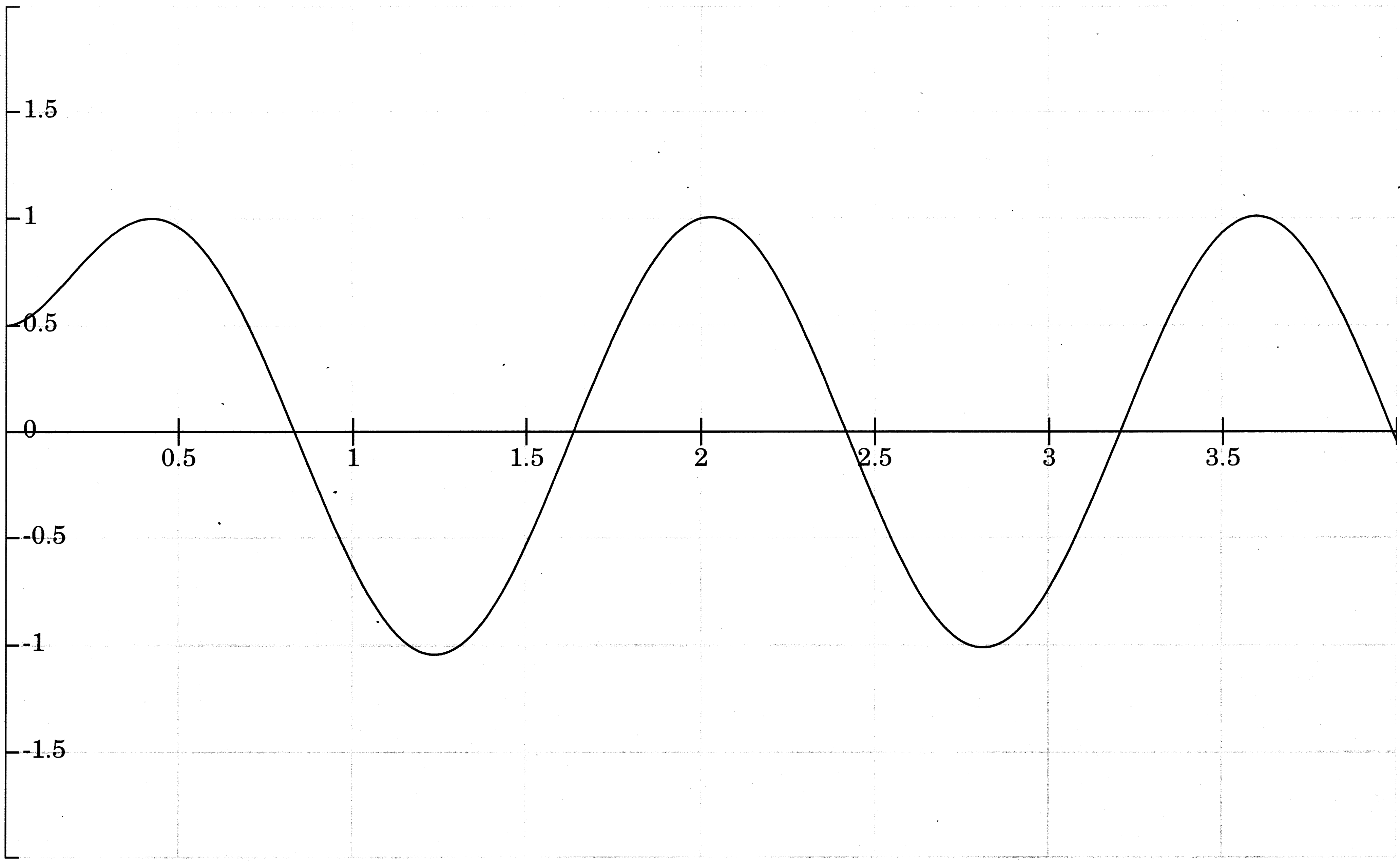
$$x'(0) = 0 \Rightarrow -3c_1 + c_2 + \frac{50}{51}(4) = 0$$

$$c_2 = -\frac{86}{51}$$

$$x(t) = \underbrace{\frac{38}{51}e^{-3t}\cos t - \frac{86}{51}e^{-3t}\sin t}_{\text{Transient Part}} \underbrace{- \frac{25}{102}\cos 4t + \frac{50}{51}\sin 4t}_{\text{Steady State Part}}$$

Transient Part

Steady State Part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2.5
3
3.5