Math 216 - Final Exam A Name key

May 7, 2014 Score ______

Show all work. Supply explanations when necessary. You must work individually on this exam.

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1. (12 points) Solve the following system of differential equations.

$$x' = x + 3y - 2t^2$$

$$y' = 3x - y + t + 5$$

$(D-1)x - 3y = -2t^2$

$-3x + (D+1)y = t+5$

$3(D-1)x - 9y = -6t^2$

$-3(D-1)x + (D^2-1)y = -t-4$

$(D^2-10)y = -6t^2-t-4$

$y'' - 10y = -6t^2-t-4$

Homo eq: $y''-10y=0$

$r^2-10=0$

$r=\sqrt{10},\ r=-\sqrt{10}$

$y(t) = c_1 e^{\sqrt{10}t} + c_2 e^{-\sqrt{10}t}$

NonHomo eq: $g(t) = -6t^2-t-4$

$y_p(t) = At^2+Bt+C$

$y_p'(t) = 2At+B$

$y_p''(t) = 2A$

$2A - 10At^2 - 10Bt - 10C = -6t^2-t-4$

$A = \frac{6}{10} = \frac{3}{5}$

$B = \frac{1}{10}$

$2A - 10C = -4 \Rightarrow \frac{6}{5}+4 = 10C$

$C = \frac{13}{25}$

$$\boxed{y(t) = c_1 e^{\sqrt{10}t} + c_2 e^{-\sqrt{10}t} + \frac{3}{5}t^2 + \frac{1}{10}t + \frac{13}{25}}$$

$$\boxed{x(t) = \frac{1}{3}(y'+y-t-5)}$$

2. (8 points) Use Lapace transforms to solve: $y' - y = 1 + te^t,\ y(0) = 0$

$$\mathcal{L}\{y'\} - \mathcal{L}\{y\} = \mathcal{L}\{1 + te^t\}.$$

$$sY(s) - 0 - Y(s) = \frac{1}{s} + \frac{1}{(s-1)^2}$$

$$Y(s)\,(s-1) = \frac{1}{s} + \frac{1}{(s-1)^2}$$

$$Y(s) = \frac{1}{s(s-1)} + \frac{1}{(s-1)^3}$$

$$\Rightarrow \boxed{y(t) = \frac{1}{2}t^2e^t + e^t - 1.}$$

3. (15 points) Use Laplace transforms to solve the system of equations.

$$x'' + 10x - 4y = 0$$

$$-4x + y'' + 4y = 0$$

$$x(0) = 0,\ x'(0) = 1,\ y(0) = 0,\ y'(0) = -1$$

$$\mathcal{L}\{x''\} + 10\,\mathcal{L}\{x\} - 4\,\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$-4\,\mathcal{L}\{x\} + \mathcal{L}\{y''\} + 4\,\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$s^2 X(s) - s\,x(0) - x'(0) + 10X(s) - 4Y(s) = 0$$

$$-4X(s) + s^2 Y(s) - s\,y(0) - y'(0) + 4Y(s) = 0$$

$$s^2 X(s) - 1 + 10X(s) - 4Y(s) = 0 \qquad (s^2+10)X(s) - 4Y(s) = 1$$

$$-4X(s) + s^2 Y(s) + 1 + 4Y(s) = 0 \qquad -4X(s) + (s^2+4)Y(s) = -1$$

Cramer's Rule

$$X(s) = \frac{s^2}{(s^2+10)(s^2+4) - 16} \Rightarrow \boxed{x(t) = \frac{\sqrt{3}}{5}\sin 2\sqrt{3}t - \frac{1}{5\sqrt{2}}\sin\sqrt{2}t}$$

$$Y(s) = \frac{-s^2-6}{(s^2+10)(s^2+4) - 16} \Rightarrow \boxed{y(t) = -\frac{\sqrt{3}}{10}\sin 2\sqrt{3}t - \frac{\sqrt{2}}{5}\sin\sqrt{2}t}$$

Math 216 - Final Exam B

May 14, 2014

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (12 points) Solve: $y' = xy^2 \cos x, \quad y(\pi) = 2$

$$\frac{dy}{y^2} = x \cos x \, dx$$

$$\boxed{y(x) = \frac{-1}{\frac{1}{2} + x \sin x + \cos x}}$$

$$\int y^{-2} \, dy = \int x \cos x \, dx$$

$+$	x	$\cos x$
$-$	1	$\sin x$
$+$	0	$-\cos x$

$$-y^{-1} = x \sin x + \cos x + C_1$$

$$y = \frac{1}{C_2 - x \sin x - \cos x}$$

$$y(\pi) = 2 \Rightarrow \frac{1}{C_2 + 1} = 2 \Rightarrow C_2 = -\frac{1}{2}$$

2. (10 points) Solve: $y'' + 2y' - 8y = 0, \quad y(0) = y'(0) = 6$

$$r^2 + 2r - 8 = 0$$

$$(r+4)(r-2) = 0$$

$$r = -4, \; r = 2$$

$$y = c_1 e^{-4t} + c_2 e^{2t}$$

$$\boxed{y(t) = e^{-4t} + 5e^{2t}}$$

$$y(0) = 6 \Rightarrow c_1 + c_2 = 6$$

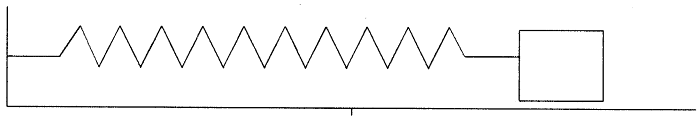
$$y'(0) = 6 \Rightarrow -4c_1 + 2c_2 = 6$$

$$4c_1 + 4c_2 = 24$$

$$-4c_1 + 2c_2 = 6$$

$$6c_2 = 30 \Rightarrow c_2 = 5, \; c_1 = 1$$

3. (20 points) A 2-kg mass is attached to a spring with spring constant 24 N/m. The damping constant for the system is 8 N-sec/m. The mass starts at the equilibrium position with an initial speed of 2 m/sec to the left. Is this mass-spring system underdamped, overdamped, or critically damped? Set up and solve the initial value problem that describes the displacement of the mass from equilibrium.



$x = 0$
(Equilibrium)

$$2x'' + 8x' + 24x = 0$$

$$x(0) = 0, \quad x'(0) = -2$$

$$r^2 + 4r + 12 = 0$$

$$r^2 + 4r + 4 = -8$$

$$(r+2)^2 = -8$$

$$r = -2 \pm 2\sqrt{2}\, i$$

$$x(t) = e^{-2t}\left(c_1 \cos 2\sqrt{2}t + c_2 \sin 2\sqrt{2}t\right)$$

$$x(0) = 0 \Rightarrow c_1 = 0$$

$$x(t) = c_2 e^{-2t} \sin 2\sqrt{2}\, t$$

$$x'(t) = -2c_2 e^{-2t} \sin 2\sqrt{2}t + 2\sqrt{2}\, c_2 e^{-2t} \cos 2\sqrt{2}t$$

$$x'(0) = -2 \Rightarrow 2\sqrt{2}\, c_2 = -2 \Rightarrow c_2 = -\frac{1}{\sqrt{2}}$$

$$\boxed{x(t) = -\frac{1}{\sqrt{2}} e^{-2t} \sin 2\sqrt{2}\, t}$$

THIS SYSTEM IS UNDERDAMPED.
(OSCILLATIONS OCCUR.)

4. (10 points) Use Euler's method with a step size of $h = 0.5$ to approximate $y(3)$, where $y(x)$ is the solution of the initial value problem $y' = xy^2$, $y(2) = 1$.

$y_0 = 1$

$x_0 = 2$

$y_1 = y_0 + h f(x_0, y_0) = 1 + 0.5(2)(1)^2 = 2$

$x_1 = 2 + 0.5 = 2.5$

$y_2 = y_1 + h f(x_1, y_1) = 2 + 0.5(2.5)(2)^2 = 7$

$x_2 = 2.5 + 0.5 = 3$

$$\boxed{y(3) \approx 7}$$

5. (5 points) For $x > 0$, let $y_1(x) = \ln x^5$ and $y_2(x) = \ln x$. Compute the Wronskian of y_1 and y_2. Briefly explain why $y(x) = c_1 y_1(x) + c_2 y_2(x)$ cannot be the general solution of a 2nd-order, linear, homogeneous differential equation.

$$W[y_1, y_2](x) = \begin{vmatrix} \ln x^5 & \ln x \\ \frac{5x^4}{x^5} & \frac{1}{x} \end{vmatrix} = \frac{\ln x^5}{x} - \frac{5x^4 \ln x}{x^5}$$

$$= \frac{5 \ln x}{x} - \frac{5 \ln x}{x} = 0$$

$y_1(x)$ & $y_2(x)$ ARE LINEARLY DEPENDENT.

A GENERAL SOL'N OF A 2^{ND} ORDER EQUATION IS A LINEAR COMBO OF LINEARLY INDEPENDENT SOL'NS.

6. (12 points) According to Newton's Law of Cooling, the temperature T at time t of an object cooling in a medium of constant temperature M is described by the differential equation

$$\frac{dT}{dt} = k(M - T),$$

where k is some constant.

(a) Solve the differential equation.

$$\frac{dT}{M-T} = k\,dt$$

$$\int \frac{dT}{M-T} = \int k\,dt$$

$$-\ln|M-T| = kt + c_1$$

$$\ln|M-T| = -kt + c_2$$

$$|M-T| = c_3 e^{-kt}$$

$$M-T = c_4 e^{-kt}$$

$$T = M - c_4 e^{-kt}$$

$$\boxed{T(t) = M + Ce^{-kt}}$$

(b) An object at 120°F is moved into a large room with an ambient temperature of 72°F. The object cools to 100°F in 6 min. Use your result from part (a) to find a formula for the temperature of the object at time t.

$T(0) = 120$

$M = 72$

$T(6) = 100$

$$T(t) = 72 + Ce^{-kt}$$

$$T(0) = 120 \Rightarrow 72 + C = 120$$

$$C = 48$$

$$T(t) = 72 + 48e^{-kt}$$

$$T(6) = 100 = 72 + 48e^{-6k}$$

$$28 = 48e^{-6k}$$

$$k = \frac{\ln\frac{28}{48}}{-6} = \frac{\ln(7/12)}{-6}$$

$$\boxed{T(t) = 72 + 48e^{\frac{\ln(7/12)}{6}t}}$$

7. (20 points) Use variation of parameters to solve the following differential equation.

$$x'' - 2x' + x = \frac{e^t}{t^2}$$

Homo eq: $x'' - 2x' + x = 0$

$r^2 - 2r + 1 = 0$

$(r-1)^2 = 0 \Rightarrow r = 1, r = 1$

$x_h(t) = c_1 e^t + c_2 t e^t$

NonHomo eq: $g(t) = \frac{e^t}{t^2}$

$x_p(t) = v_1(t) e^t + v_2(t) t e^t$

$$W(t) = \begin{vmatrix} e^t & t e^t \\ e^t & e^t + t e^t \end{vmatrix}$$

$$= e^{2t}$$

$$v_1(t) = \int \frac{-\frac{e^t}{t^2} t e^t}{e^{2t}} dt = \int -\frac{1}{t} dt$$

$$= -\ln|t|$$

$$v_2(t) = \int \frac{\frac{e^t}{t^2} e^t}{e^{2t}} dt = \int \frac{1}{t^2} dt$$

$$= -\frac{1}{t}$$

$x_p(t) = -\ln|t| e^t - e^t$

$$X(t) = X_h(t) + X_p(t) = c_3 e^t + c_2 t e^t - e^t \ln|t|$$

8. (10 points) Solve: $\overbrace{(xy-1)}^{M(x,y)}\,dx + \overbrace{(x^2-xy)}^{N(x,y)}\,dy = 0$

$$\frac{\partial M}{\partial y} = x \qquad \frac{\partial N}{\partial x} = (2x-y) \qquad \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{y-x}{x(x-y)} = -\frac{1}{x}$$

$$\mu(x) = e^{\int -\frac{1}{x}\,dx} = e^{-\ln|x|} = \frac{1}{x}, \quad x > 0$$

$$\left(y - \frac{1}{x}\right)dx + (x-y)\,dy = 0$$

$$\frac{\partial F}{\partial x} = y - \frac{1}{x} \Rightarrow F(x,y) = xy - \ln x + g(y)$$

$$\frac{\partial F}{\partial y} = x - y \Rightarrow F(x,y) = xy - \frac{1}{2}y^2 + h(x)$$

$$F(x,y) = xy - \ln x - \frac{1}{2}y^2$$

SOL'N IS

$$\boxed{xy - \ln x - \frac{1}{2}y^2 = C, \quad x > 0}$$

9. (10 points) Find the orthogonal trajectories for the family of curves described by the equation $Cy^2 = x^3$.

$$C = \frac{x^3}{y^2}, \qquad 2Cy\frac{dy}{dx} = 3x^2$$

$$\frac{2x^3}{y}\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{3y}{2x}$$

ORTHO TRAJ'S SATISFY

$$\frac{dy}{dx} = -\frac{2x}{3y}$$

$$3y\,dy = -2x\,dx$$

$$\frac{3}{2}y^2 = -x^2 + C$$

$$\boxed{x^2 + \frac{3}{2}y^2 = C}$$

10. (6 points) Given below are the differential equations or the equations of motion of some mass-spring systems. Each describes exactly one of the following situations: *simple harmonic motion*, *underdamped motion*, *overdamped motion*, or *critically damped motion*. Match each equation with the corresponding situation.

(a) $x(t) = 2e^{-2t} + 5te^{-2t}$

CHAR EQ HAS REPEATED ROOT $\Rightarrow$ CRITICALLY DAMPED

(b) $x'' + 8x' + 17x = 0$

$b^2 - 4ac = 64 - 4(17) < 0 \Rightarrow$ UNDERDAMPED

(c) $x(t) = \sqrt{6}\sin(4t + \pi)$

NO DAMPING $\Rightarrow$ SIMPLE HARMONIC MOTION

(d) $2x'' + 5x' + 3x = 0$

$b^2 - 4ac = 25 - 4(2)(3) = 1 > 0$

$\Rightarrow$ OVERDAMPED