A Closer Look at Bobo's Sequence

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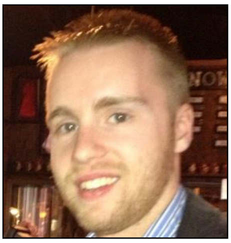

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E. Ray Bobo considered the sequence $\{a(n)\}_{n=2}^{\infty}$, where $a(n)$ is the least positive integer such that

$$\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{a(n)} > 1$$

in a Classroom Capsule in this JOURNAL [**1**]. He showed that $a(n)$ takes one of three values, $\lfloor ne \rfloor - 2$, $\lfloor ne \rfloor - 1$, or $\lfloor ne \rfloor$, where $\lfloor \cdot \rfloor$ denotes the least integer/integer floor function. He took special interest in the n for which $a(n) = \lfloor ne \rfloor$. Let

$$\begin{aligned} \mathcal{B} &= \{n : a(n) = \lfloor ne \rfloor\} \\ &= \{4, 11, 18, 25, 32, 36, 43, 50, 57, 64, 71, 75, 82, 89, 96, 103, 114, \\ &\qquad 121, 128, 135, 142, 146, 153, 160, 167, 174, 185, 192, 199, \ldots\}. \end{aligned}$$

We refer to the elements of $\mathcal{B}$ as the *Bobo numbers*.

While Bobo did not rule out the possibility that $a(n) = \lfloor ne \rfloor - 2$, his numerical experiments (through $n = 2115$) yielded no examples. Furthermore, in his analysis of $\mathcal{B}$, Bobo observed that the gaps between consecutive elements formed an intriguing pattern of 4s, 7s, and 11s. At the end of his article, he posed several questions, two of which we consider here:

(A) Does the pattern in the gaps of $\mathcal{B}$ persist, or does chaos eventually take over?

(B) Is $a(n) = \lfloor ne \rfloor - 2$ impossible?

http://dx.doi.org/10.4169/college.math.j.45.3.199
MSC: 11Y55, 26D15

In *The On-Line Encyclopedia of Integer Sequences* [**3**], the sequence A103762 is essentially $\{a(n)\}_{n=2}^{\infty}$. The posted comments hint at the same questions as those posed by Bobo. For the most part, we answer these questions using elementary methods only. In fact, a central theme will be the application of standard integral approximation techniques encountered in beginning calculus.

Ruling out a case

We begin with Question (B): Can $a(n) = \lfloor ne \rfloor - 2$? For a positive integer $n \geq 2$, we apply the midpoint rule for numerical integration using subintervals of length one to approximate $\int_{n-\frac{1}{2}}^{\lfloor ne \rfloor - \frac{3}{2}} \frac{1}{x}\,dx$ (see Figure 1). Because the graph of $y = \frac{1}{x}$ is concave up in the first quadrant, the midpoint rule underestimates the integral. It follows that

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{\lfloor ne \rfloor - 2} < \int_{n-\frac{1}{2}}^{\lfloor ne \rfloor - \frac{3}{2}} \frac{1}{x}\,dx = \ln\left(\frac{\lfloor ne \rfloor - \frac{3}{2}}{n - \frac{1}{2}}\right).$$

In turn,

$$\ln\left(\frac{\lfloor ne \rfloor - \frac{3}{2}}{n - \frac{1}{2}}\right) \leq \ln\left(\frac{ne - \frac{3}{2}}{n - \frac{1}{2}}\right) < \ln\left(\frac{ne - \frac{e}{2}}{n - \frac{1}{2}}\right) = 1.$$

Putting together these inequalities, we have

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{[ne] - 2} < 1$$

whenever $n \geq 2$; therefore, $a(n) = \lfloor ne \rfloor - 2$ is indeed impossible.

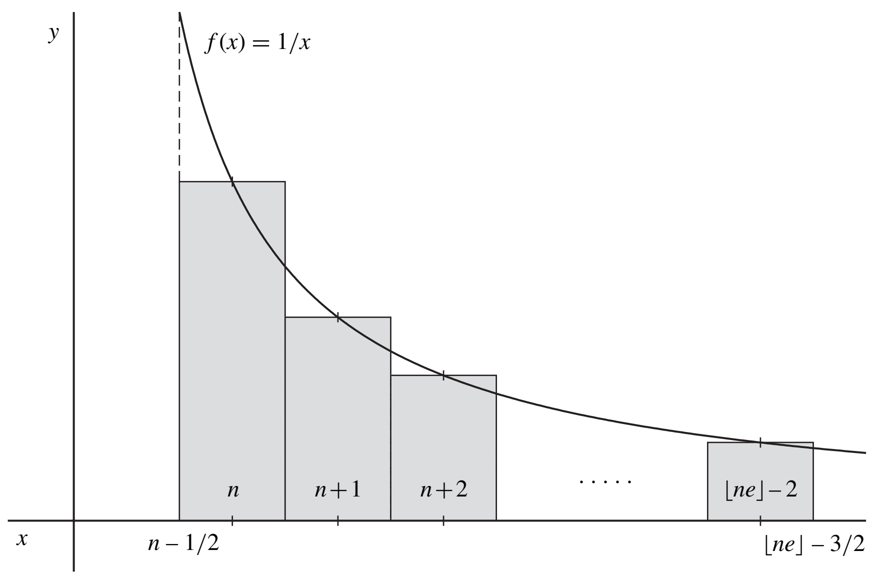

Figure 1. The midpoint rule approximation of $\frac{1}{x}$.

There remain only the two possibilities, $a(n) = \lfloor ne \rfloor - 1$ or $a(n) = \lfloor ne \rfloor$. Numerical experiments indicate that both occur frequently, but $a(n) = \lfloor ne \rfloor$ for only about $\frac{1}{7}$ of the positive integers. The fraction $\frac{1}{7}$ is probably a slight overestimate, but we are getting ahead of ourselves.

Fractional parts must have the right size

Turning to Question (A), our goal is to determine the possible gaps in $\mathcal{B}$ and estimate their frequencies. We begin by finding a lower bound for

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{\lfloor ne \rfloor - 1},$$

which provides a clue for when $a(n) = \lfloor ne \rfloor$ (i.e., when n is a Bobo number). A useful lower bound is found by approximating $\int_n^{\lfloor ne \rfloor} \frac{1}{x}\, dx$ with a left-endpoint Riemann sum over subintervals of length one:

$$\int_n^{\lfloor ne \rfloor} \frac{1}{x}\, dx = \ln\left(\frac{\lfloor ne \rfloor}{n}\right) < \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{\lfloor ne \rfloor - 1}. \tag{1}$$

From a table of values of $\ln\left(\frac{\lfloor ne \rfloor}{n}\right)$, we observe that this lower bound is especially small (and hence $a(n)$ is more likely equal to $\lfloor ne \rfloor$ than $\lfloor ne \rfloor - 1$) whenever $\lfloor ne \rfloor$ and its predecessor, $\lfloor (n-1)e \rfloor$, are as close together as possible. Writing $\{x\}$ for the fractional part of x, we have

$$\lfloor ne \rfloor = \lfloor (n-1)e + e \rfloor = \lfloor (n-1)e \rfloor + 2 + \lfloor \{(n-1)e\} + \{e\} \rfloor.$$

Recognizing that $\lfloor \{(n-1)e\} + \{e\} \rfloor$ can only be 0 or 1, we can rewrite this as

$$\lfloor ne \rfloor = \begin{cases} \lfloor (n-1)e \rfloor + 2 & \text{if } \{(n-1)e\} + \{e\} < 1, \\ \lfloor (n-1)e \rfloor + 3 & \text{if } \{(n-1)e\} + \{e\} \geq 1. \end{cases} \tag{2}$$

Notice that $\lfloor ne \rfloor$ and $\lfloor (n-1)e \rfloor$ are as close as possible in the first case, precisely when $\{(n-1)e\} < 1 - \{e\} \approx 0.2817$. Using (2), it is routine to show that the inequality $\{(n-1)e\} < 1 - \{e\}$ is a necessary and sufficient condition to guarantee

$$\ln\left(\frac{\lfloor ne \rfloor}{n}\right) < \ln\left(\frac{\lfloor (n-1)e \rfloor}{n-1}\right).$$

Therefore, the smallness of the lower bound in (1) coincides with the smallness of $\{(n-1)e\}$. However, the inequality $\{(n-1)e\} < 1 - \{e\}$ does not guarantee that n is a Bobo number. Indeed, if we define $\mathcal{S}$ by

$$\begin{aligned} \mathcal{S} &= \{n : \{(n-1)e\} < 1 - \{e\}\} \\ &= \{4, 8, 11, 15, 18, 22, 25, 29, 32, 36, 40, 43, 47, 50, 54, 57, \\ &\qquad 61, 64, 68, 71, 75, 79, 82, 86, 89, \ldots\}, \end{aligned}$$

we see that $\mathcal{S}$ contains elements that $\mathcal{B}$ does not. On the other hand, we show next that $\mathcal{B} \subset \mathcal{S}$. Our proof requires the following technical lemma.

Lemma. *For any positive integer n,*

$$\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{\lfloor (n-1)e\rfloor+2}>1.$$

Proof. The lemma follows by using the trapezoidal rule with subintervals of length one to approximate $\int_n^{\lfloor (n-1)e\rfloor+2}\frac{1}{x}\,dx$. This gives

$$\frac{1}{n}+\cdots+\frac{1}{\lfloor (n-1)e\rfloor+2}>\ln\left(\frac{\lfloor (n-1)e\rfloor+2}{n}\right)+\frac{1}{2n}+\frac{1}{2(\lfloor (n-1)e\rfloor+2)}.$$

The right-hand side of this inequality is greater than

$$\ln\left(\frac{(n-1)e+1}{n}\right)+\frac{1}{2n}+\frac{1}{2ne},$$

which decreases for $n\geq 9$ and has limit 1. So the lemma holds when $n\geq 9$ and it can be verified directly for $n=1,\ldots,8$. ■

Proposition 1. *With $\mathcal{B}$ and $\mathcal{S}$ as defined above, $\mathcal{B}\subset\mathcal{S}$. In other words, if n is a Bobo number, then $\{(n-1)e\}<1-\{e\}\approx 0.2817$.*

Proof. Suppose that n is a Bobo number. By definition, $a(n)=\lfloor ne\rfloor$ and it follows that

$$\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{\lfloor ne\rfloor-1}<1.$$

By (2), we know that $\lfloor ne\rfloor-1$ is either $\lfloor (n-1)e\rfloor+1$ or $\lfloor (n-1)e\rfloor+2$. By the lemma, the latter is impossible. So $\lfloor ne\rfloor-1=\lfloor (n-1)e\rfloor+1$ and, again from (2), we see that $\{(n-1)e\}<1-\{e\}$. ■

Proposition 1 sheds some light on the structure of the set of Bobo numbers, but is not sufficient to provide a satisfying answer to Question (A). Before more progress can be made, another important observation is required: In order for n to be a Bobo number, $\{(n-1)e\}$ must be small, but not too small. The next proposition quantifies this idea.

Proposition 2. *For $x\geq 1$, let*

$$f(x)=xe+1-e-x\exp\left(1-\frac{1}{2x}-\frac{1}{2xe-2}\right).$$

If n is a Bobo number, then $\{(n-1)e\}>f(n)$. Specifically, for any $n\in\mathcal{B}$,

$$\{(n-1)e\}>11-4e\approx 0.1269.$$

Proof. Suppose that n is a Bobo number. By the definition and the trapezoidal rule approximation of $\int_n^{\lfloor ne\rfloor-1}\frac{1}{x}\,dx$ with subintervals of length one, it follows that

$$1>\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{\lfloor ne\rfloor-1}>\ln\left(\frac{\lfloor ne\rfloor-1}{n}\right)+\frac{1}{2n}+\frac{1}{2\lfloor ne\rfloor-2}.$$

From this and the fact that

$$\ln\left(\frac{\lfloor ne\rfloor-1}{n}\right)+\frac{1}{2n}+\frac{1}{2\lfloor ne\rfloor-2}>\ln\left(\frac{\lfloor (n-1)e\rfloor+1}{n}\right)+\frac{1}{2n}+\frac{1}{2ne-2},$$

we have

$$1 - \frac{1}{2n} - \frac{1}{2ne - 2} > \ln\left(\frac{\lfloor (n-1)e \rfloor + 1}{n}\right).$$

It follows that

$$n \exp\left(1 - \frac{1}{2n} - \frac{1}{2ne - 2}\right) > \lfloor (n-1)e \rfloor + 1 = (n-1)e + 1 - \{(n-1)e\},$$

which establishes the first part of the proposition. The second part can be verified directly for the Bobo numbers 4, 11, 18, 25, 32, and 36. For Bobo numbers greater than 36, the second part follows from the fact that f is an increasing function and that $f(36) > 11 - 4e$. ■

We now take our first big step toward answering Question (A).

Theorem 1. *Suppose that n is a Bobo number and that k is a positive integer. If*

$$0.1548 \approx 3e - 8 \leq \{ke\} \leq 9 - 3e \approx 0.8452,$$

then $n + k$ is not a Bobo number.

Proof. The proof requires only our two propositions and some arithmetic with fractional parts. Suppose that n is a Bobo number. Since $1 - \{e\} = 3 - e$, it follows from Propositions 1 and 2 that

$$11 - 4e < \{(n-1)e\} < 3 - e.$$

With the given conditions on k, this inequality can be combined with the inequality in the theorem statement to give

$$(11 - 4e) + (3e - 8) < \{(n-1)e\} + \{ke\} < (3 - e) + (9 - 3e)$$

or

$$3 - e < \{(n-1)e\} + \{ke\} < 12 - 4e. \tag{3}$$

At this point, there are two cases. If $\{(n-1)e\} + \{ke\} < 1$, then it follows that $\{(n-1)e\} + \{ke\} = \{(n+k-1)e\}$ and inequality (3) becomes

$$3 - e < \{(n+k-1)e\} < 1.$$

Therefore, $n + k$ cannot be a Bobo number by Proposition 1.

If instead $\{(n-1)e\} + \{ke\} \geq 1$, then $\{(n-1)e\} + \{ke\} = \{(n+k-1)e\} + 1$. In this case, inequality (3) becomes

$$1 \leq \{(n+k-1)e\} + 1 < 12 - 4e$$

and $n + k$ cannot be a Bobo number by Proposition 2. ■

Theorem 1 directly rules out an infinity of gap sizes between Bobo numbers and it applies equally to nonconsecutive Bobo numbers. For the time being, we are especially interested that it rules out certain gaps between consecutive Bobo numbers, namely those of size 1, 2, 3, 5, 6, 8, 9, or 10. Next, we show that consecutive Bobo numbers must differ by no more than 11. It will follow that the gaps between consecutive Bobo numbers are restricted to size 4, 7, or 11.

Gaps cannot exceed 11

Propositions 1 and 2 place necessary conditions on Bobo numbers. It is time for a sufficient condition. This will lead to our next major theorem.

Proposition 3. *Suppose that n is an integer and $n \geq 2$. If $\{ne\} > \frac{1}{2}(e-1)$, then n is a Bobo number.*

Proof. Using the midpoint rule with subintervals of length one to approximate $\int_{n-\frac{1}{2}}^{\lfloor ne\rfloor-\frac{1}{2}} \frac{1}{x}\,dx$, we have

$$\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{\lfloor ne\rfloor-1} < \ln\left(\frac{\lfloor ne\rfloor-\frac{1}{2}}{n-\frac{1}{2}}\right) = \ln\left(e+\frac{\frac{1}{2}(e-1)-\{ne\}}{n-\frac{1}{2}}\right).$$

If $\{ne\} > \frac{1}{2}(e-1)$, then the right-hand side is less than $\ln e = 1$. This indicates that n must be a Bobo number. ■

With Proposition 3, we can show that for any positive integer n, at least one of the integers $n+1, n+2, \ldots, n+11$ is a Bobo number. To do so, we will describe conditions on n that make each of $\{(n+1)e\}, \{(n+2)e\}, \ldots, \{(n+11)e\}$ exceed $\frac{1}{2}(e-1)$. The following proposition gives an example of our approach.

Proposition 4. *Let $L = \frac{1}{2}(e-1) \approx 0.8591$. If n is a positive integer satisfying*

$$0.5494 \approx L - \{6e\} < \{ne\} < 1 - \{6e\} \approx 0.6903,$$

then $n+6$ is a Bobo number.

Proof. Since $\{6e\}$ is slightly less than 0.3097, the conditions on $\{ne\}$ imply that $\{ne\} + \{6e\} = \{(n+6)e\}$. Therefore, after adding $\{6e\}$ across the inequality in the proposition, we have $L < \{(n+6)e\} < 1$. By Proposition 3, we conclude that $n+6$ must be a Bobo number. ■

Proposition 4 is summarized in row 7 of Table 1. The additional ten results in the table, corresponding to ten similar propositions, can be verified in the same way. Notice that the intervals described in Table 1 cover [0, 1], with some overlap. This leads to the following theorem.

Theorem 2. *For any positive integer n, there is a $k = 1, 2, \ldots, 11$ such that $(n+k)$ is a Bobo number. As a consequence, gaps between consecutive Bobo numbers cannot exceed 11.*

Proof. Since $\{ne\}$ lies somewhere in [0, 1], it corresponds to at least one row of Table 1. Any such row gives a Bobo number that differs from n by no more than 11. ■

Frequencies of gaps

Taken together, Theorems 1 and 2 establish that 4, 7, or 11 are the only possible gaps between consecutive Bobo numbers. This provides a partial answer to Question (A), but we can say much more. For example, Theorem 1 rules out gaps of size 15 between nonconsecutive Bobo numbers. Therefore, consecutive gaps of 4 and 11, in either order, are impossible. Similarly, consecutive 4-gaps and consecutive 11-gaps are impossible. Theorem 1 also rules out gaps of size 42, so consecutive 7-gaps are limited to

Table 1. Conditions on n that guarantee $n + k$ is a Bobo number, using $L = \frac{1}{2}(e - 1)$.

Row	Interval	Implication
1	$[0, 1 - \{4e\}) \cup (1 + L - \{4e\}, 1] \approx [0, 0.1269) \cup (0.9860, 1]$	$n + 4 \in \mathcal{B}$
2	$(L - \{8e\}, 1 - \{8e\}) \approx (0.1129, 0.2537)$	$n + 8 \in \mathcal{B}$
3	$(L - \{e\}, 1 - \{e\}) \approx (0.1409, 0.2817)$	$n + 1 \in \mathcal{B}$
4	$(L - \{5e\}, 1 - \{5e\}) \approx (0.2677, 0.4086)$	$n + 5 \in \mathcal{B}$
5	$(L - \{9e\}, 1 - \{9e\}) \approx (0.3946, 0.5355)$	$n + 9 \in \mathcal{B}$
6	$(L - \{2e\}, 1 - \{2e\}) \approx (0.4226, 0.5634)$	$n + 2 \in \mathcal{B}$
7	$(L - \{6e\}, 1 - \{6e\}) \approx (0.5494, 0.6903)$	$n + 6 \in \mathcal{B}$
8	$(L - \{10e\}, 1 - \{10e\}) \approx (0.6763, 0.8172)$	$n + 10 \in \mathcal{B}$
9	$(L - \{3e\}, 1 - \{3e\}) \approx (0.7043, 0.8452)$	$n + 3 \in \mathcal{B}$
10	$(L - \{7e\}, 1 - \{7e\}) \approx (0.8312, 0.9720)$	$n + 7 \in \mathcal{B}$
11	$[0, 1 - \{11e\}) \cup (1 + L - \{11e\}, 1] \approx [0, 0.09890) \cup (0.9580, 1]$	$n + 11 \in \mathcal{B}$

no more than five (five consecutive 7-gaps begin with 36). It is clear that the 4-7-11 pattern observed by Bobo continues. But what about the frequencies of these gaps?

Since e is irrational, the terms of the sequence $\{\{ne\}\}_{n=1}^{\infty}$ are uniformly distributed in the interval $(0, 1)$ (see [**2**] for an elementary proof). As a consequence, the inequality in Proposition 3,

$$\{ne\} > \frac{1}{2}(e - 1) \approx 0.8591, \tag{4}$$

is satisfied by roughly 14.09% of the integers in any set of the form $\{1, 2, \ldots, N\}$ for sufficiently large N. While the converse of Proposition 3 is not true—the Bobo number 36 is a counterexample—it is almost true in the following sense. Using Propositions 1 and 2, we can show that when n is a Bobo number,

$$\{ne\} = \{(n - 1)e\} + \{e\} > f(n) + \{e\} = f(n) + e - 2, \tag{5}$$

where f is defined in Proposition 2. Since the numbers $\{ne\}$ are uniformly distributed in $(0, 1)$ and $\lim_{n\to\infty} (f(n) + e - 2) = \frac{1}{2}(e - 1)$, most Bobo numbers will eventually satisfy (4). It follows that about 14.09% of the positive integers are Bobo numbers. As mentioned earlier, this is a bit less than $\frac{1}{7}$ and agrees closely with numerical experiments (see Table 2).

Table 2. Frequencies of Bobo numbers less than or equal to 5,000,000.

Description	Count	Relative Frequency
Bobo numbers	704,298	14.086%
Bobo numbers preceding 4-gaps	69,935	1.399%
Bobo numbers preceding 7-gaps	564,434	11.289%
Bobo numbers preceding 11-gaps	69,929	1.399%

Table 1 provides an algorithm for generating Bobo numbers: Choose a positive integer n, compute $\{ne\}$, and use the table to obtain a Bobo number greater than n. While this algorithm produces Bobo numbers, it is not guaranteed to produce every Bobo number. Because Table 1 is based on Proposition 3, the algorithm cannot be expected to generate Bobo numbers (such as 36) that fail to satisfy (4). However, as described above, these anomalous Bobo numbers are rare, let us call them *exceptional*. With this in mind, we will disregard the exceptional Bobo numbers and estimate the frequencies of the 4-7-11 gaps.

According to Table 1, a gap of size 4 from n to a Bobo number is predicted when $\{ne\}$ lies in the union of intervals given in row 1. If n is a Bobo number, then it satisfies (5). This rules out the inclusion of $\{ne\}$ in the interval $[0, 0.1269)$. Therefore, a gap of 4 from one Bobo number to the next is predicted only when $\{ne\}$ is between 0.9860 and 1. Since this interval has length 0.014, we should expect that roughly 1.4% of the positive integers are Bobo numbers preceding a gap of 4.

To predict a gap of 7, we use similar reasoning, focusing on row 10 of Table 1. A gap of 7 from an unexceptional Bobo number, n, to the next Bobo number is predicted when n satisfies (4) and $\{ne\} \in (0.8312, 0.9720)$. In other words, $\{ne\}$ lies between 0.8591 and 0.9720, an interval of length 0.1129. It follows that roughly 11.29% of the positive integers are Bobo numbers preceding a gap of 7.

A gap of 11 is predicted by using row 11 in Table 1. Reasoning as above, a gap of this size from one Bobo number, n, to another will occur when $\{ne\} \in (0.9580, 1]$. However, we must rule out the 11-gaps that occur as a result of consecutive 4s and 7s. By carefully examining the intervals in rows 1, 10, and 11, we see that a gap of size 11 alone is predicted when $\{ne\}$ is between 0.9720 and 0.9860. Since this interval has length 0.014, we should expect that roughly 1.4% of the positive integers are Bobo numbers preceding a gap of 11.

The frequencies estimated above agree closely with numerical experiments given in Table 2. In response to Question (A), it appears that chaos does *not* eventually take over. Nonetheless, an exact formula for the nth Bobo number is, alas, beyond our reach.

Acknowledgment. This article is the result of an undergraduate research project undertaken by the younger author. We would like to thank the referees, the previous editor, and the current editor for their thoughtful analysis and helpful suggestions. We are also grateful for the advice from Natalie McGathey.

Summary. Any sum of reciprocals of consecutive natural numbers must eventually exceed 1. The final term of such a sum is a function of the initial term. In a 1995 Classroom Capsule, E. Ray Bobo described some properties of that function and posed several questions regarding its possible values. We answer some of those questions, primarily using integral approximations from calculus.

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