A remarkably elementary proof of the irrationality of $e$
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The standard proofs of the irrationality of $e$ make use of the infinite series representation

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$  \hspace{1cm} (1)

or the corresponding alternating series representation for $1/e$. (One such proof is given at the end of this article.) While these proofs are elementary, they obviously require some familiarity with infinite series. The following proof requires only integration-by-parts and some basic properties of the Riemann integral. The sum (1) follows as a consequence, thereby making this proof useful as an introduction to infinite series.

$e$ is irrational.

Proof: Suppose $e = a/b$, where $a$ and $b$ are positive integers. Choose an integer $n \geq \max\{b, e\}$. Now consider the definite integral $\int_0^1 e^{-x} \, dx$. This integral is easily evaluated to give $1 - \frac{1}{e}$. On the other hand, repeated integration-by-parts ($n$ times) gives

$$1 - \frac{1}{e} = \frac{1}{e} \left( 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \right) + \int_0^1 \frac{x^n}{n!} e^{-x} \, dx.$$  

Upon multiplying both sides by $e$ and isolating the integral, we obtain

$$e - 1 - \left( 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \right) = e \int_0^1 \frac{x^n}{n!} e^{-x} \, dx.$$  \hspace{1cm} (2)

Multiplying both sides of (2) by $n!$ gives

$$n!(e - 1) - n! \left( 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \right) = e \int_0^1 x^n e^{-x} \, dx.$$  

Because of the choice of $n$ and the assumption that $e$ is rational, the left hand side must reduce to an integer. However the value of the expression on the right is between zero and one. Indeed

$$0 < e \int_0^1 x^n e^{-x} \, dx \leq e \int_0^1 x^n \, dx = \frac{e}{n+1} < 1.$$  

This contradiction implies that $e$ must be irrational. \diamond

Notice that the integral in (2) approaches zero as $n \to \infty$. Therefore we obtain (1) as a by-product of the proof. The series representation (1) was derived in a similar way by Chamberland in [1] and by Johnson in [2].

A proof using the series for $1/e$ ...

Use the fact that

$$e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!},$$  

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and let \( S_n \) denote the \( n \)th partial sum of the series:

\[
1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}.
\]

Notice that \( S_n \) is a rational number, and it can be written in the form \( M/n! \), where \( M \) is an integer. By the alternating series estimation theorem, it follows that

\[
S_n - \frac{1}{(n+1)!} < e^{-1} < S_n \quad \text{for even } n
\]

and

\[
S_n < e^{-1} < S_n + \frac{1}{(n+1)!} \quad \text{for odd } n.
\]

In either case, \( e^{-1} \) is strictly between two rational numbers of the forms \( \frac{a}{(n+1)!} \) and \( \frac{a+1}{(n+1)!} \), where \( a \) is an integer. It follows that \( e^{-1} \) cannot be written as a fraction with denominator \( (n+1)! \) for any \( n \geq 0 \). Since any rational number can be written as a fraction with denominator \( (n+1)! \), we conclude that \( e^{-1} \) cannot be a rational number. Since \( 1/e \) is irrational, it follows that \( e \) is irrational. (This proof is similar to Sondow’s geometric proof [3].)

References

