Some Sums via Integration by Parts

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The familiar integration by parts formula

$$\int u\,dv = uv - \int v\,du$$

is usually used to simplify integrals. However, when it is used to complicate integrals, interesting things sometimes happen. For instance, Taylor's theorem is easily proved by starting with the simple equation

$$f(x) = f(a) + \int_a^x f'(t)\,dt,$$

choosing $u = f'(t)$ and $dv = dt$, obtaining $du = f''(t)\,dt$ and $v = -(x-t)$, and then repeatedly integrating by parts. The integrals get messier, but the payoff is well worth it.

In [1] and [2] repeated integration by parts is used in a similar way to obtain the series expansion

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

The purpose of this note is to describe how several other series expansions can be obtained by integration.

As a first example, we consider the constant $\ln 2$ and the definite integral

$$\int_0^1 \frac{x}{1+x^2}\,dx = \frac{\ln 2}{2}.$$

This integral is easily evaluated by means of a simple substitution. However, repeated integration by parts, starting with $u = (1+x^2)^{-1}$ and $dv = x\,dx$, could be used to obtain an infinite series that converges to $\ln 2/2$.

Instead of repeatedly integrating, we can streamline the approach by defining the sequence $\{a_n\}_{n=1}^{\infty}$ where

$$a_n = \int_0^1 \frac{x^{2n-1}}{(1+x^2)^n}\,dx.$$

We have already seen that $a_1 = \ln 2/2$ and, upon integrating by parts, we have

$$a_n = \frac{1}{n2^{n+1}} + a_{n+1}.$$

Repeated use of this formula gives

$$\ln 2/2 = \left(\sum_{n=1}^{K} \frac{1}{n2^{n+1}} \right) + a_{K+1}$$

or

$$\ln 2 = \left(\sum_{n=1}^{K} \frac{1}{n2^{n}} \right) + 2a_{K+1}.$$

It is not difficult to show that $a_k \to 0$ as $k \to \infty$. Therefore we obtain the result

$$\ln 2 = \lim_{K\to\infty} \left(\sum_{n=1}^{K} \frac{1}{n2^{n}} \right) = \sum_{n=1}^{\infty} \frac{1}{n2^{n}}.$$

A number of other series expansions can be obtained in the same way. Some of these (as well as the previous example) are summarized below.

1. $\ln 2 = \sum_{n=1}^{\infty} \frac{1}{n2^n}$
   - Integral: $\int_0^1 \frac{x}{1+x^2}\,dx = \frac{\ln 2}{2} = a_1$
   - Sequence: $a_n = \int_0^1 \frac{x^{2n-1}}{(1+x^2)^n}\,dx$
   - Recurrence: $a_n = \frac{1}{n2^{n+1}} + a_{n+1}, \quad n = 1, 2, 3, \ldots$
2. $\ln 2 = \frac{1}{2} + \sum_{n=0}^{\infty} \frac{1}{2^n(n+3)(n+2)(n+1)}$
   - Integral: $\int_0^1 x \ln(1+x)\,dx = \frac{1}{4} = \frac{\ln 2}{2} + a_1$
   - Sequence: $a_n = \frac{-1}{n(n+1)} \int_0^1 \frac{x^{n+1}}{(1+x)^n}\,dx$
   - Recurrence: $a_n = \frac{-1}{2^n n(n+1)(n+2)} + a_{n+1}, \quad n = 1, 2, 3, \ldots$
3. $\phi - 1 = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2 5^n}$, where $\phi = (1+\sqrt{5})/2$

- Integral: $\displaystyle\int_0^1 (4x+1)^{-3/2}\,dx = \frac{\phi - 1}{\sqrt{5}} = a_1$
- Sequence: $\displaystyle a_n = \frac{2^{n-1}(2n-1)!!}{(n-1)!}\int_0^1 \frac{x^{n-1}}{(4x+1)^{(2n+1)/2}}\,dx$
- Recurrence: $\displaystyle a_n = \frac{2^{n-1}(2n-1)!!}{n!5^n\sqrt{5}} + a_{n+1}, \quad n = 1, 2, 3, \ldots$

4. $\displaystyle\frac{\pi}{2} = \sum_{n=0}^{\infty} \frac{n!}{(2n+1)!!}$

   - Integral: $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx = \frac{\pi}{4} = \frac{1}{2} + a_1$
   - Sequence: $\displaystyle a_n = \frac{n!2^n}{(2n-1)!!}\int_0^1 \frac{x^{2n}}{(1+x^2)^{n+1}}\,dx$
   - Recurrence: $\displaystyle a_n = \frac{n!}{2(2n+1)!!} + a_{n+1}, \quad n = 1, 2, 3, \ldots$

5. $\displaystyle\frac{\pi}{\sqrt{3}} + \ln 2 = \sum_{n=1}^{\infty} \frac{3^n(n-1)!}{2^n \cdot 1 \cdot 4 \cdot 7 \cdots (3n-2)}$

   - Integral: $\displaystyle\int_0^1 \frac{1}{1+x^3}\,dx = \frac{1}{9}\left(\sqrt{3}\pi + 3\ln 2\right) = \frac{1}{2} + a_1$
   - Sequence: $\displaystyle a_n = \frac{3^{n-1}(n-1)!}{1 \cdot 4 \cdot 7 \cdots (3n-5)}\int_0^1 \frac{x^{3n-3}}{(1+x^3)^n}\,dx$
   - Recurrence: $\displaystyle a_n = \frac{3^{n-1}(n-1)!}{2^n \cdot 1 \cdot 4 \cdot 7 \cdots (3n-2)} + a_{n+1}, \quad n = 1, 2, 3, \ldots$

6. $\displaystyle 1 = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

   - Integral: $\displaystyle\int_0^1 1\,dx = 1 = a_1$
   - Sequence: $\displaystyle a_n = \frac{1}{n}\int_0^1 \frac{1}{x^n}\,x^n\,dx$ (Integrate with $u = x^{-n}$ and $dv = x^n\,dx$)
   - Recurrence: $\displaystyle a_n = \frac{1}{n(n+1)} + a_{n+1}, \quad n = 1, 2, 3, \ldots$

Each of these sums could be obtained by using a more traditional approach. However, the approach outlined here provides an interesting application of integration by parts. In addition, it could be used as an engaging introduction to sequences and series.

References

[1] M. Chamberland, *The series for e via integration*, College Mathematics Journal, 30 (1999), p. 397.

[2] W. Johnson, *Power series without Taylor's theorem*, American Mathematical Monthly, 91 (1984), pp. 367–369.