There's More Than One Way to Integrate that Function

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Abstract

Are trig substitutions obsolete? How do you integrate by parts? What's the easy way to integrate $e^x \cos(x)$? In this session, you will see some integration tips and techniques that are not usually discussed in typical calculus textbooks. The focus is on general techniques rather than clever tricks.

Guiding principles

When solving problems, modern math students are encouraged to use common sense approaches, think creatively, and welcome nontraditional methods.

Nevertheless, in standard calculus texts, the integration techniques are pretty traditional.

Goals

1. Discuss a number of general tips and techniques for integration that are not typically found in calculus textbooks.
2. Discuss the pros and cons of teaching these less traditional techniques.
3. Discuss the cognitive obstacles students encounter when integrating, and discuss how less traditional techniques can be helpful.

Opening problem

Consider the following integral:

$$\int (1 + 2x^2)\, e^{x^2}\, dx$$

This integral was also the opening problem in Kirthi Premadasa's presentation "Intriguing Integrals Arising out of 'Impossible' Ones" (WisMATYC 2017).

Opening problem—Possible solution

1. Write $\int (1+2x^2)e^{x^2}\,dx = \int e^{x^2}\,dx + \int x \cdot 2xe^{x^2}\,dx$.
2. Focus on the last integral and integrate by parts:

$$u = x,\ dv = 2xe^{x^2}\,dx \quad \longrightarrow \quad du = dx,\ v = e^{x^2}.$$

3. $\int (1+2x^2)e^{x^2}\,dx = \int e^{x^2}\,dx + xe^{x^2} - \int e^{x^2}\,dx.$
4. $\int (1+2x^2)e^{x^2}\,dx = xe^{x^2} + const.$

Opening problem—Possible solution (cont.)

More generally, the opening problem has a very special form:

$$\int [f(x) + x \cdot f'(x)]dx = x \cdot f(x) + const.$$

Of course, with a small change to the problem, the simple closed form disappears.

For example, $\int (1 + 3x^2)e^{x^2}\,dx$ has no elementary closed form.

“Impossible” integrals

The existence of “unsolvable” integrals is a source of frustration for students.

- They give up too easily, believing their integral is unsolvable.
- They do not know how to recognize unsolvable integrals.
- Why bother trying if the integral may be unsolvable?

Undetermined coefficients

I prefer to solve the opening problem by recognizing that

$$\int (1 + 2x^2)e^{x^2}\,dx = (Ax + B)e^{x^2} + const,$$

provided a closed form exists.

Now differentiate the right-hand side and solve for the undetermined coefficients.

Why use the method of undetermined coefficients?

- Unsolvable integrals can be identified (provided the correct form has been chosen).
- Students are already familiar with pattern-matching techniques (e.g., factoring, conic sections, partial fractions).
- Students will use undetermined coefficients in their futures (e.g., partial fractions, differential equations, transform methods).
- Being able to recognize the form of a solution is valuable skill.
- Computer algebra systems use it.

Another example for undetermined coefficients

$$\int x^4 e^{5x}\,dx = e^{5x}(Ax^4 + Bx^3 + Cx^2 + Dx + E) + const$$

Differentiate and solve for the undetermined coefficients.

$$\begin{array}{ccccccccccc} 5A & & & & & & & & & = & 1 \\ 4A & + & 5B & & & & & & & = & 0 \\ & & 3B & + & 5C & & & & & = & 0 \\ & & & & 2C & + & 5D & & & = & 0 \\ & & & & & & D & + & 5E & = & 0 \end{array}$$

This bidiagonal system is easily solved to give

$$A = 1/5, \quad B = -4/25, \quad C = 12/125, \quad D = -24/625, \quad E = 24/3125.$$

Another example for undetermined coefficients

There are lots of ways to evaluate $\int e^x \cos x \, dx$.

- repeated integration by parts
- use $e^{ix} = \cos x + i \sin x$
- transform methods, etc.

But it is very easy to use undetermined coefficients...

$$\int e^x \cos x \, dx = Ae^x \cos x + Be^x \sin x + const$$

Integration by parts

All calculus textbooks describe the integration by parts formula,

$$\int u\,dv = uv - \int v\,du,$$

which follows from the product rule for differentiation.

Integration by parts (cont.)

Most textbooks discuss the challenge of wisely choosing u and dv.

- u must be easy to differentiate.
- v must be easily obtained from dv.
- Ideally, $v\,du$ must be simpler than $u\,dv$.

Few textbooks offer the suggestion of choosing u according to LIATE (or LIPTE).

Integration by parts (cont.)

A good general rule of thumb...

When using the integration by parts formula, choose u as the type of function that appears first in LIATE.

- L - Logarithmic
- I - Inverse trigonometric
- A - Algebraic (P - Polynomial or Power)
- T - Trigonometric
- E - Exponential

Integration by parts (cont.)

LIATE examples...

1. $\displaystyle\int 3x^2 e^x \, dx$

   Choose $u = 3x^2$ and $dv = e^x \, dx$

2. $\displaystyle\int x \sin^{-1} x \, dx$

   Choose $u = \sin^{-1} x$ and $dv = x \, dx$

3. $\displaystyle\int \ln x \, dx$

   Choose $u = \ln x$ and $dv = dx$

Tabular integration by parts

The tabular method of integration by parts is underrated!

It is most often used for problems requiring repeated integration by parts that end with a zero derivative.

Tabular integration by parts (cont.)

Evaluate $\int x^2 \cos x \, dx$.

signs		u and derivs		dv/dx and antiderivs
$+$	$\longrightarrow$	x^2	$\searrow$	$\cos x$
$-$	$\longrightarrow$	$2x$	$\searrow$	$\sin x$
$+$	$\longrightarrow$	2	$\searrow$	$-\cos x$
$-$		0		$-\sin x$

$$\int x^2 \cos x \, dx = x^2 \sin x + 2x \cos x - 2 \sin x + const$$

Tabular integration by parts (cont.)

The tabular method can be used with any integration-by-parts problem as a nice way to organize work.

$$\int x^3 \ln x \, dx = \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^3 \, dx$$

signs	u and derivs	dv/dx and antiderivs
$+$	$\ln x$	x^3
$-$	$\frac{1}{x}$	$\frac{1}{4}x^4$

Tabular integration by parts (cont.)

One more example: $\int \ln x \, dx = x \ln x - \frac{x}{2} - \frac{x}{6} - \frac{x}{12} - \frac{x}{20} - \cdots$

signs	u and derivs	dv/dx and antiderivs
$+$	$\ln x$	1
$-$	$1/x$	x
$+$	$-1/x^2$	$x^2/2$
$-$	$2/x^3$	$x^3/6$
$\vdots$	$\vdots$	$x^4/24$
$\vdots$	$\vdots$	$\vdots$

Quotient rule integration by parts

Starting with the quotient rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{du}{v} - \frac{u\,dv}{v^2},$$

it follows that

$$\int \frac{u\,dv}{v^2} = \int \frac{du}{v} - \frac{u}{v} \qquad (1)$$

or

$$\int \frac{du}{v} = \frac{u}{v} - \int \frac{u\,dv}{v^2}. \qquad (2)$$

Quotient rule integration by parts may be gaining popularity [2, 3].

Quotient rule integration by parts (cont.)

The quotient rule integration by parts formulas can be very challenging to use, but they are worth discussing.

- They put the product and quotient rules on equal footing when it comes to antidifferentiation.
- Problems arise that are perfect for the formulas.

Quotient rule integration by parts (cont.)

Here is an exercise from the Larson textbook: $\displaystyle\int \frac{xe^{2x}}{(2x+1)^2}\,dx.$

The form of the problem suggests trying (1) with

$$v = 2x+1, \quad dv = 2\,dx, \quad u = \frac{1}{2}xe^{2x}.$$

This makes $du = (x+\frac{1}{2})e^{2x}$ and

$$\int \frac{xe^{2x}}{(2x+1)^2}\,dx = \int \frac{(x+\frac{1}{2})e^{2x}}{2x+1}\,dx - \frac{\frac{1}{2}xe^{2x}}{2x+1}$$

$$= \frac{1}{2}\int e^{2x}\,dx - \frac{xe^{2x}}{4x+2}.$$

Quotient rule integration by parts (cont.)

Here is an example taken from [3]: $\displaystyle\int \frac{\sin(x^{-1/2})}{x^2}\,dx.$

Use formula (2) with

$$v = x^{1/2} \text{ and } du = \frac{\sin(x^{-1/2})}{x^{3/2}}.$$

The details are left for you.

Quotient rule integration by parts (cont.)

Here are three more examples for which formula (1) works nicely.

1. $\displaystyle\int \frac{1}{x(\ln x)^2}\,dx$

2. $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx$

3. $\displaystyle\int \frac{8}{(4x^2+1)^2}\,dx$

Quotient rule integration by parts (cont.)

Some questions to consider...

1. Are there LIATE-type guidelines?
2. How would a tabular method look?
3. Repeated QR integration by parts?
4. Are there problems involving recurring integrals?

Partial Fractions 1 - Cover-up method

When decomposing into partial fractions, Heaviside's cover-up method is quick, easy, and typically ignored.

1. Cover up the highest power of a linear factor.
2. Substitute the zero of the covered factor.
3. Evaluate to find the corresponding coefficient.

Partial Fractions 1 (cont.)

Example: $$\frac{8x+7}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}.$$

Cover up $(x-1)^2$: $$\frac{8x+7}{\cancel{(x-1)^2}(x-2)}$$

Plug in $x = 1$: $$\frac{8+7}{\cancel{(x-1)^2}(1-2)} = -15$$

$B = -15$

Partial Fractions 1 (cont.)

Example: $$\frac{8x+7}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}.$$

Cover up $(x-2)$: $$\frac{8x+7}{(x-1)^2\cancel{(x-2)}}$$

Plug in $x=2$: $$\frac{16+7}{(2-1)^2\cancel{(x-2)}} = 23$$

$C = 23$

Unfortunately, the method does not directly help to find A.

Partial Fractions 2 - Limit method

The limit method is essentially a standard method with the added feature of having ∞ available as a convenient value for substitution. I have never seen the limit method described in a calculus textbook.

1. Multiply by a well-chosen factor.
2. Take a limit to make terms vanish.

Partial Fractions 2 (cont.)

Example: $$\frac{8x+7}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}.$$

Multiply by $(x-1)^2$: $$\frac{8x+7}{x-2} = A(x-1) + B + \frac{C(x-1)^2}{x-2}.$$

Let $x \to 1$: $$\frac{15}{-1} = B$$

$B = -15$

Partial Fractions 2 (cont.)

Example: $$\frac{8x+7}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}.$$

Multiply by $x-2$: $$\frac{8x+7}{(x-1)^2} = \frac{A(x-2)}{x-1} + \frac{B(x-2)}{(x-1)^2} + C.$$

Let $x \to 2$: $$\frac{23}{1} = C$$

$C = 23$

Partial Fractions 2 (cont.)

Example: $$\frac{8x+7}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}.$$

Multiply by x: $$\frac{8x^2+7x}{(x-1)^2(x-2)} = \frac{Ax}{x-1} + \frac{Bx}{(x-1)^2} + \frac{Cx}{x-2}.$$

Let $x \to \infty$: $0 = A + C$

$A = -23$

Partial Fractions

Incidentally, even though we often think of $\int \sec^3 x\,dx$ as being a classic integration by parts problem, it is a wonderful partial fractions example. (See [4] for more details.)

$$\int sec^3 x\,dx = \int \frac{\cos x}{\cos^4 x}\,dx = \int \frac{\cos x}{(1-\sin^2 x)^2}\,dx = \int \frac{1}{(1-u^2)^2}\,du,$$

where $u = \sin x$.

Trigonometric substitutions

Trigonometric substitutions are encountered without fail in any thorough discussion of integration techniques. Does it have to be that way?

In a recent CMJ article [1], the authors envision the demise of trig substitutions. They argue that the types of integrals handled with trig substitutions can be treated with other substitutions.

Trigonometric substitutions (cont.)

Suppose

$$R = (a^2 - x^2)^{1/2}, \quad (a^2 + x^2)^{1/2}, \quad \text{or} \quad (x^2 - a^2)^{1/2}.$$

Consider the common trig sub integrals

$$\int x^n R^k \, dx, \quad \int \frac{x^n}{R^k} \, dx, \quad \int \frac{R^k}{x^n} \, dx, \quad \int \frac{1}{x^n R^k} \, dx,$$

where $n \geq 0$, $k > 0$ are integers and k is odd.

These integrals can be evaluated using

$$u^2 = R^2 \text{ for } n \text{ odd} \quad \text{or} \quad u^2 = \frac{R^2}{x^2} \text{ for } n \text{ even}.$$

Trigonometric substitutions (cont.)

Example: $\displaystyle\int \frac{1}{x^2\sqrt{9-x^2}}\,dx.$

Assume $u, x > 0$ and let $u^2 = (9-x^2)/x^2$ so that

$$u^2 = 9x^{-2} - 1, \quad u^2x^2 = 9 - x^2, \quad 2u\,du = -18x^{-3}\,dx.$$

Substitute to get

$$\int \frac{1}{x^2\sqrt{9-x^2}}\,dx = -\frac{1}{9}\int du.$$

Trigonometric substitutions (cont.)

Example: $\displaystyle\int \frac{x^3}{\sqrt{x^2+4}}\,dx.$

Assume $u, x > 0$ and let $u^2 = x^2 + 4$ so that

$$2u\,du = 2x\,dx, \quad x^2 = u^2 - 4.$$

Substitute to get

$$\int \frac{x^3}{\sqrt{x^2+4}}\,dx = \int (u^2 - 4)\,du.$$

Trigonometric substitutions (cont.)

I like trig substitutions! But my students clearly prefer making the u^2-substitutions, especially in the case when n is odd.

The u^2-substitutions require cleverness and algebraic skills, but my students claim they are easier than trig substitutions.

Sometimes the u^2-substitutions are easier, but they can be quite challenging!

Triple integration

I close by mentioning a recent triple integration article that was of interest to my students.

Withers [5] describes a procedure for evaluating triple integrals that does not require visualizing the region of integration. As part of the MAA's "Most Read Collection," the article is freely available until December 2018 at http://explore.tandfonline.com/content/est/MAA-most-read-2018.

Shameless plug: I have an article coauthored by students that is also freely available at the same site.

Thanks for coming!

Please feel free to contact me.

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