Uncommon Approaches to Some Common Calculus Problems

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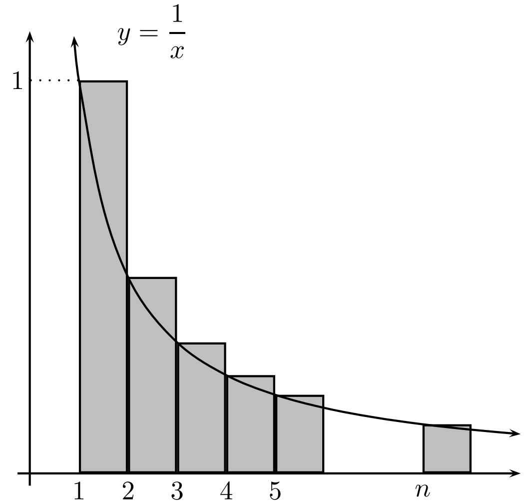
Most of the common problems of mathematics have tried-and-true solution methods that have stood the test of time. On occasion, however, a more unusual approach to a problem may be useful for shedding light on a specific concept. In this presentation we will look at some uncommon, and perhaps unfamiliar, approaches to solving a number of common calculus problems.

1 The harmonic series diverges

The harmonic series,

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots,$$

is probably the most familiar example of a divergent series whose terms tend to zero. There are two very popular approaches to establishing its divergence. One approach is based on comparing $\sum_{k=1}^{n} 1/k$ and $\int_1^{n+1} 1/x\,dx$. Along these lines, there are the integral test and the proof suggested by the following figure:

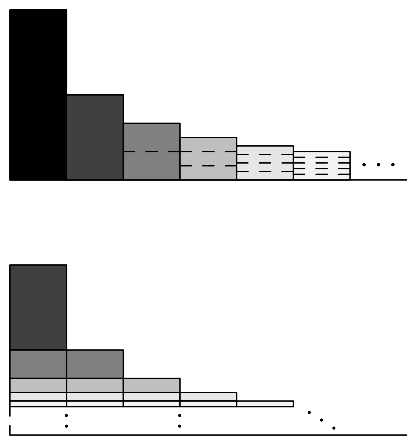


$$\int_1^{n+1} \frac{dx}{x} = \ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n} = H_n$$

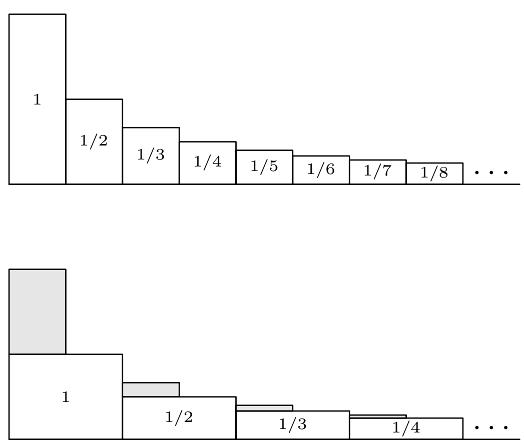
The second popular approach is to use a condensation argument similar to the classical proof given by Nicole Oresme (circa 1350):

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n} &= 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)++\left(\frac{1}{9}+\cdots+\frac{1}{16}\right)\cdots\\ &> 1+\frac{1}{2}+\frac{2}{4}+\frac{4}{8}+\frac{8}{16}+\cdots\\ &= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots\end{aligned}$$

Some more unusual proofs establish divergence by regrouping terms in order to show that the harmonic series is greater than itself. The following proof without words was given by Jim Belk.[1] It is the visual equivalent of Johann Bernoulli's proof (see [2]).



This "proof" is the visual equivalent of a divergence proof given by Leonard Gillman [5].



For more proofs of divergence of the harmonic series, see [6] and [7].

[1] http://cornellmath.wordpress.com/2007/07/12/harmonic-digression/

2 Euler's formula

Students encounter Euler's formula, $e^{ix} = \cos x + i \sin x$, when they study 2nd-order, linear, constant-coefficient differential equations. The most popular approach to proving this formula makes use of the power series expansions of e^x, $\cos x$, and $\sin x$. An alternative approach takes advantage of a fact that DE students should know: 1st-order, linear, initial value problems have unique solutions. Euler's formula can be established by showing that $y = e^{ix}$ and $y = \cos x + i \sin x$ both satisfy the IVP

$$\frac{dy}{dx} = iy, \quad y(0) = 1.$$

3 Some integrals using Euler's formula

The integration formulas

$$\int e^x \cos x\, dx = \frac{e^x}{2}(\cos x + \sin x) + C$$

and

$$\int e^x \sin x\, dx = \frac{e^x}{2}(\sin x - \cos x) + C$$

are usually derived by using integration by parts. Students familiar with Euler's formula can derive both formulas at once without using integration by parts. Ignoring the constants of integration, we have

$$\begin{aligned}
\int e^x \cos x\, dx + i \int e^x \sin x\, dx &= \int e^x e^{ix}\, dx = \int e^{(1+i)x}\, dx \\
&= \left(\frac{1}{1+i}\right) e^{(1+i)x} \quad = \quad \left(\frac{1-i}{2}\right) e^x e^{ix} \\
&= \frac{e^x}{2}(1-i)(\cos x + i \sin x) \\
&= \frac{e^x}{2}(\cos x + \sin x) \quad + \quad i\,\frac{e^x}{2}(-\cos x + \sin x)
\end{aligned}$$

4 The natural log and powers of x

In many calculus textbooks the natural logarithm is defined by means of a definite integral:

$$\ln x = \int_1^x \frac{1}{t}\, dt, \quad x > 0.$$

Since the power rule,

$$\int t^n\, dt = \frac{1}{n+1} t^{n+1} + C,$$

does not apply when $n = -1$, the integral definition of the natural log seems to isolate $\ln x$ from the power functions. The integral definition also does little to help students understand why $\ln x$ grows so slowly. To help with these issues, Finlayson [4] suggests that the identity

$$\int_1^x t^{k-1}\, dt = \frac{x^k - 1}{k}$$

leads naturally to a limit definition of the log:

$$\ln x = \lim_{k\to 0} \frac{x^k - 1}{k}.$$

This limit puts $\ln x$ in its place among the powers of x. With this limit in mind, one would expect the log to grow slower than any power of x.

It also turns out that many of the properties of the natural log can be derived from this limit. For example, assuming $a, b > 0$, we have

$$\begin{aligned}
\ln ab &= \lim_{k\to 0} \frac{(ab)^k - 1}{k} \quad = \quad \lim_{k\to 0} \frac{a^k b^k - 1}{k} \\
&= \lim_{k\to 0} \frac{a^k b^k - b^k + b^k - 1}{k} \\
&= \lim_{k\to 0} \frac{b^k(a^k - 1)}{k} + \lim_{k\to 0} \frac{b^k - 1}{k} \\
&= \left(\lim_{k\to 0} b^k\right)\left(\lim_{k\to 0} \frac{a^k - 1}{k}\right) + \lim_{k\to 0} \frac{b^k - 1}{k} \\
&= (1)(\ln a) + \ln b \quad = \quad \ln a + \ln b
\end{aligned}$$

The derivative of $\ln x$ is easy to derive if you are willing to replace $f(h) = (x+h)^k$ with its standard linear approximation at $h = 0$, $L(h) = x^k + hkx^{k-1}$:

$$\begin{aligned}
\frac{d}{dx} \ln x &= \lim_{h\to 0} \frac{1}{h}\left(\lim_{k\to 0} \frac{(x+h)^k - 1}{k} - \lim_{k\to 0} \frac{x^k - 1}{k}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\lim_{k\to 0} \frac{(x+h)^k - x^k}{k}\right) \\
&= \lim_{h\to 0} \left(\lim_{k\to 0} \frac{(x+h)^k - x^k}{hk}\right) \\
&= \lim_{h\to 0} \left(\lim_{k\to 0} \frac{x^k + hkx^{k-1} - x^k}{hk}\right) \\
&= \lim_{h\to 0} \left(\lim_{k\to 0} x^{k-1}\right) \quad = \quad x^{-1}.
\end{aligned}$$

Finally, the limit definition of the log can help put its inverse function in its own place among the powers of x. The limit definition of e^x,

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n,$$

can be derived by taking the limit of the inverse of

$$y = \frac{x^k - 1}{k}.$$

5 An alternative to integrating by parts

Normally evaluating $\int x^n e^{ax}\, dx$ is an exercise in repeated integration by parts. An alternative approach introduces students to the method of undetermined coefficients.

With a bit of experience differentiating, most students would expect an antiderivative of $x^n e^{ax}$ to have the form $e^{ax}p_n(x)$, where p_n is a polynomial of degree n. Therefore, $\int x^n e^{ax}\,dx$ can be evaluated by differentiating and equating coefficients. For example, to evaluate $\int x^4 e^{5x}\,dx$, we assume an antiderivative of the form

$$e^{5x}(Ax^4 + Bx^3 + Cx^2 + Dx + E)$$

and differentiate to obtain

$$5e^{5x}(Ax^4 + Bx^3 + Cx^2 + Dx + E) + e^{5x}(4Ax^3 + 3Bx^2 + 2Cx + D) = x^4 e^{5x}.$$

Equating coefficients leads to the following system of equations:

$$\begin{array}{cccccccccccc}
5A & & & & & & & & & = & 1 \\
4A & + & 5B & & & & & & & = & 0 \\
 & & 3B & + & 5C & & & & & = & 0 \\
 & & & & 2C & + & 5D & & & = & 0 \\
 & & & & & & D & + & 5E & = & 0
\end{array}$$

This bidiagonal system is easily solved to give

$$A = 1/5, \qquad B = -4/25, \qquad C = 12/125, \qquad D = -24/625, \qquad E = 24/3125,$$

and therefore

$$\int x^4 e^{5x}\,dx = e^{5x}(\frac{1}{5}x^4 - \frac{4}{25}x^3 + \frac{12}{125}x^2 - \frac{24}{625}x + \frac{24}{3125}) + C.$$

6 Exact DEs and integration by parts

Exact differential equations such as

$$(2xy - 9x^2)dx + (2y + x^2 + 1)dy = 0$$

are often solved by partial integration. Students who misunderstand the solution method sometimes "solve" the equation as follows:

$$\begin{aligned}
\int (2xy - 9x^2)\,dx + \int (2y + x^2 + 1)\,dy &= C \\
x^2y - 3x^3 \quad + \quad y^2 + x^2y + y \quad &= \quad C,
\end{aligned}$$

where the integrals are evaluated partially and added. It is often difficult to convince students that this method is wrong. However, the method is perfectly correct if we compute total integrals, using integration by parts for mixed-variable integrals:

$$(2xy - 9x^2)dx + (2y + x^2 + 1)dy = 0$$

$$\int 2xy\,dx - \int 9x^2\,dx + \int 2y\,dy + \int x^2\,dy + \int 1\,dy = C.$$

The first and fourth integrals require integration by parts, but in reality, only one of them must be evaluated. Letting $u = y$ and $dv = 2x\,dx$, we have

$$\int 2xy\,dx = x^2 y - \int x^2\,dy.$$

Substituting this into the equation above, we obtain the correct solution

$$x^2 y - \int x^2\,dy - 3x^3 + y^2 + \int x^2\,dy + y = C$$

or

$$x^2 y - 3x^3 + y^2 + y = C.$$

Unfortunately, the integration-by-parts approach cannot always be used to solve exact DEs. Phelps gives an excellent analysis of the method in [10].

7 Geometric series

There are a number of ways to find the sum of a geometric series. This uncommon approach uses a telescoping series.

$$a - ar + ar - ar^2 + ar^2 - ar^3 + \cdots - ar^{n-1} + ar^{n-1} - ar^n = a - ar^n$$

$$a(1-r) + ar(1-r) + ar^2(1-r) + \cdots + ar^{n-1}(1-r) = a - ar^n$$

$$\sum_{k=0}^{n-1} ar^k = \frac{a - ar^n}{1-r}$$

If $\lim_{n\to\infty} r^n = 0$, we have

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

8 Distance from point to plane

Consider these common point/plane problems:

1. Find the distance from a given point to a given plane.

2. Find the point on a given plane that is closest to a given point.

Typically the first problem is solved by using a vector projection, and the second is an optimization problem. The following vector approach solves both problems simultaneously.

Suppose we are given the point $P\,(x_0, y_0, z_0)$, and we wish to find both the distance to the plane $ax + by + cz = d$ and the point on the plane closest to P. The equation

$$(x\hat{\imath} + y\hat{\jmath} + z\hat{k}) = (x_0\hat{\imath} + y_0\hat{\jmath} + z_0\hat{k}) + t(a\hat{\imath} + b\hat{\jmath} + c\hat{k})$$

defines a set of parametric equations for the line passing through P, perpendicular to the plane. If we normalize the plane's normal vector, we obtain

$$(x\hat{\imath} + y\hat{\jmath} + z\hat{k}) = (x_0\hat{\imath} + y_0\hat{\jmath} + z_0\hat{k}) + t\frac{(a\hat{\imath} + b\hat{\jmath} + c\hat{k})}{\sqrt{a^2 + b^2 + c^2}},$$

where the parameter t now represents the signed distance from P to the point (x, y, z) in the direction of $a\hat{\imath} + b\hat{\jmath} + c\hat{k}$. At this point, substituting the parametric equations for x, y, and z into the plane's equation, $ax + by + cz = d$, will allow us to solve for the distance t. We can then evaluate the parametric equations at that t to find the corresponding xyz-point.

9 A sum via integration by parts

The improper integral, $\int_1^0 \ln x \, dx = 1$, is easily evaluated using standard integration by parts (and L'Hôpital's rule). Students familiar with tabular integration by parts may be tempted to try this approach. Unfortunately, most would believe that the tabular method has failed them, and they would probably give up as the table became more complicated. However, a bit of perseverance would pay off and give them an unexpected introduction to infinite series.

signs	u and du/dx	dv/dx and $\int dv$
$+$	$\ln x$	1
$-$	$1/x$	x
$+$	$-1/x^2$	$x^2/2$
$-$	$2/x^3$	$x^3/6$
$+$	$-6/x^4$	$x^4/24$
$\vdots$	$\vdots$	$x^5/120$
$\vdots$	$\vdots$	$\vdots$

From the table, we find that

$$\int_1^0 \ln x \, dx = x \ln x - \frac{x}{2} - \frac{x}{6} - \frac{x}{12} - \frac{x}{20} - \cdots \Big|_1^0$$

and after evaluating at $x = 0$ and $x = 1$, we have

$$1 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots .$$

10 Synthetic division and Horner's method

In algebra classes, synthetic division is usually described as a trimmed-down or collapsed version of long division. Students learn later, via the remainder theorem, that synthetic division can be used to evaluate polynomials. Alternatively, one can show directly how synthetic division is equivalent to Horner's method for evaluating polynomials.

For example, suppose we wish to evaluate $p(2)$ when $p(x) = 3x^3 + 2x^2 - 5x + 2$. After rewriting $p(x) = x(x(x(3) + 2) - 5) + 2$, it is easy to see how the evaluation of $p(2)$ corresponds to the synthetic division table below:

	3	2	-5	2
2		6	16	22
	3	8	11	24

The numbers in the final row of the table come from within the consecutive nested parentheses. Specifically, the 8 arises from $2(3) + 2$, the 11 arises from $2(8) - 5$, and the 24 arises from $2(11) + 2$.

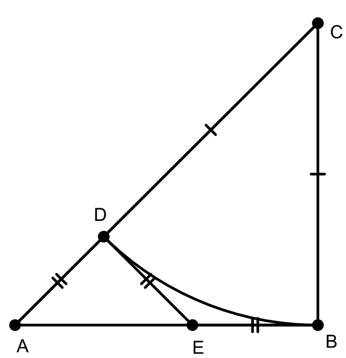
It has been shown that Horner's method is optimal, in the sense that any method for evaluating a polynomial requires at least as many operations. And, incidentally, Horner's method predates Horner by well over a thousand years.

11 $\sqrt{2}$ is irrational

There are a number of clever ways to show that $\sqrt{2}$ is irrational. Here are two very nice, unusual approaches.

Proof #1: (This proof is similar to one given by Niven [8].) Let $f(x) = (\sqrt{2}-1)x$. Suppose $\sqrt{2}$ is rational and let m be the least positive integer such that $\sqrt{2}\,m$ is an integer. It follows that $f(m)$ is a positive integer and, since $(\sqrt{2} - 1) < 1$, we must have $f(m) < m$. Now notice that $\sqrt{2}f(m) = (2 - \sqrt{2})m$, and this must be an integer. So, $f(m)$ is a positive integer less than m with the property that $\sqrt{2}f(m)$ is an integer. Based on the definition of m, we have a contradiction.

Proof #2: (This proof was given by Apostol [1].) If $\sqrt{2}$ is rational, then there must exist a least positive integer m such that $\sqrt{2}\,m$ is an integer. Therefore, there must be a smallest isosceles right triangle with integer side lengths. However, a geometric argument, suggested by the figure below, shows that for any isosceles right triangle with integer sides lengths, there exists a smaller one with the same property. (If $\triangle ABC$ is an isosceles right triangle with integer side lengths, then so is $\triangle EDA$.)



12 Exponential decay

While most of us lack the resources to carry out exponential decay experiments with radioactive materials and Geiger counters, we often overlook cheap alternatives. Dice and random-number generators make excellent substitutes for radioactive isotopes. For example, a large group of dice can be used to simulate decay in which one-sixth of an amount decays each time period—simply toss the dice, remove all the sixes, and repeat. The following Python program[2] can be used to simulate the decay of a radioactive isotope with a user-supplied half-life.

```
import random, math

ipop = 5000   # Initial amount
thalf = 4     # Half-life
time = 20     # How long will the experiment last?

num = math.exp( - math.log(2) / thalf )
pop = ipop
print "Time:", 0, " Decayed:", 0, " Remaining:", ipop
for i in range(time):
   count = 0
   for j in range(pop):
      x = random.random()
      if x < num:
         count += 1
   print "Time:", i+1, " Decayed:", ipop-count, " Remaining:", count
   pop = count
```

13 Monte Carlo methods

In the mid-1940's, John von Neuman and Stan Ulam developed the Monte Carlo method as a way of approximating solutions of unmanageable problems by mimicking random processes. Named one of the top-ten algorithms of the 20th century, the Monte Carlo method has a place in our calculus classrooms. For example, Monte Carlo integration can be used to approximate the area under the graph of the positive-valued function f on $[a, b]$:

1. Choose a number M greater than the maximum of f on $[a, b]$.
2. Generate N random points (x, y), where $x \in [a, b]$ and $y \in [0, M]$.
3. Let P be the number of points satisfying $y \leq f(x)$.
4. The area under the graph is approximately $(P/N) \cdot M(b - a)$.

[2]Python is a free, open source programming language. You can run this program online at www.codepad.org.

Here is a Python program for approximating $\int_0^{\pi} \sin x \, dx$:

```
import random, math

def f(x):
   return math.sin(x)

a = 0.0
b = math.pi
M = 1.0
num = 1000   # Number of random points

count = 0
for i in range(num):
   x = random.uniform(a,b)
   y = random.uniform(0,M)
   if y <= f(x):
      count += 1
print "Approximate value is ", (float(count) / num) * M * (b-a)
```

Monte Carlo integration can be especially useful for approximating the values of double and triple integrals.

14 Another way to sum a series

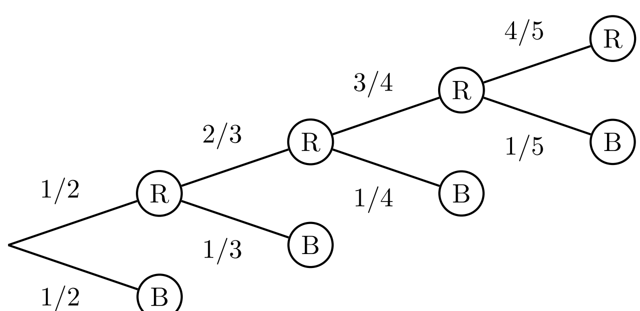
This approach to summing the series

$$\sum_{i=1}^{\infty} \frac{1}{i(i+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots$$

was presented by Pfaff and Tran [9]. We start by considering the following game:

> A red marble and a blue marble are placed into an urn. A marble is selected at random. If the marble is blue, you win. Otherwise, replace the red marble, add another red marble, and repeat the process until you win.

This tree diagram shows the probabilities associated with the first few stages of the game.



Let B_n be the event of drawing a blue marble (i.e. winning) on the nth draw. It follows that the probability of winning (eventually) is given by

$$P(B_1) + P(B_2) + P(B_3) + P(B_4) + \cdots \quad = \quad \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots.$$

Now let R_n be the event of drawing n consecutive red marbles, and notice that $P(R_n) = 1/(n+1)$. Since $P(R_n) \to 0$, the events B_i exhaust the sample space. So you must eventually draw a blue marble, and the sum above must converge to 1.

Incidentally, it is easy to show that the expected number of draws required to win is given by $\sum_{n=2}^{\infty} 1/n$. Since the harmonic series diverges, this is a game you will win, but it should take forever.

15 $\sum$ 1/p diverges

In 1938 Paul Erdős gave a pair of very clever proofs of the divergence of the sum of the reciprocals of the primes [3]. His second proof is the simpler of the two, but it is also the less familiar.

The proof makes use of the following ideas:

(i) $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

(ii) We denote the primes, in ascending order, by $p_1, p_2, p_3, p_4, \ldots$.

(iii) Let N be a given positive integer. For any positive integer m, there are $\lfloor N/m \rfloor$ integers between 1 and N that are divisible by m.

Begin by using the fact that

$$\sum_{i=2}^{\infty} \frac{1}{i(i+1)} = \sum_{i=2}^{\infty} \left(\frac{1}{i} - \frac{1}{i+1} \right) = \frac{1}{2}$$

to establish that

$$\sum_{i=1}^{\infty} \frac{1}{p_i^2} < \frac{1}{4} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}.$$

Now assume that $\displaystyle\sum_{i=1}^{\infty} \frac{1}{p_i}$ converges. It follows that there exists an integer K such that

$$\sum_{i=K+1}^{\infty} \frac{1}{p_i} < \frac{1}{8}.$$

Call $p_{K+1}, p_{K+2}, p_{K+3}, \ldots$ the "large primes," and $p_1, p_2, \ldots, p_K$ the "small primes."

Let N be a positive integer and let $y \leq N$ be a positive, squarefree integer with only small prime divisors. The integer y must have the factorization

$$y = p_1^{m_1} p_2^{m_2} p_3^{m_3} \cdots p_K^{m_K},$$

where each exponent m_i has value 0 or 1. It follows from the multiplication principle that there are 2^K possible choices for the integer y. Those 2^k integers must remain after we remove from the sequence $1, 2, 3, \ldots, N$ all those integers that are not squarefree or have large prime divisors. Therefore, we must have the following inequality:

$$2^K \geq N - \sum_{i=1}^{K} \left\lfloor \frac{N}{p_i^2} \right\rfloor - \sum_{i=K+1}^{\infty} \left\lfloor \frac{N}{p_i} \right\rfloor \geq N - \sum_{i=1}^{K} \frac{N}{p_i^2} - \sum_{i=K+1}^{\infty} \frac{N}{p_i} > N - \frac{3}{4}N - \frac{1}{8}N = \frac{N}{8}.$$

However, if we simply choose $N \geq 2^{K+3}$, we have a contradiction.

References

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