Uncommon Approaches to Some Common Calculus Problems

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Most of the common problems of mathematics have tried-and-true solution methods that have stood the test of time. On occasion, however, a more unusual approach to a problem may be useful for shedding light on a specific concept. In this presentation we will look at some uncommon, and perhaps unfamiliar, approaches to solving a number of common calculus problems.

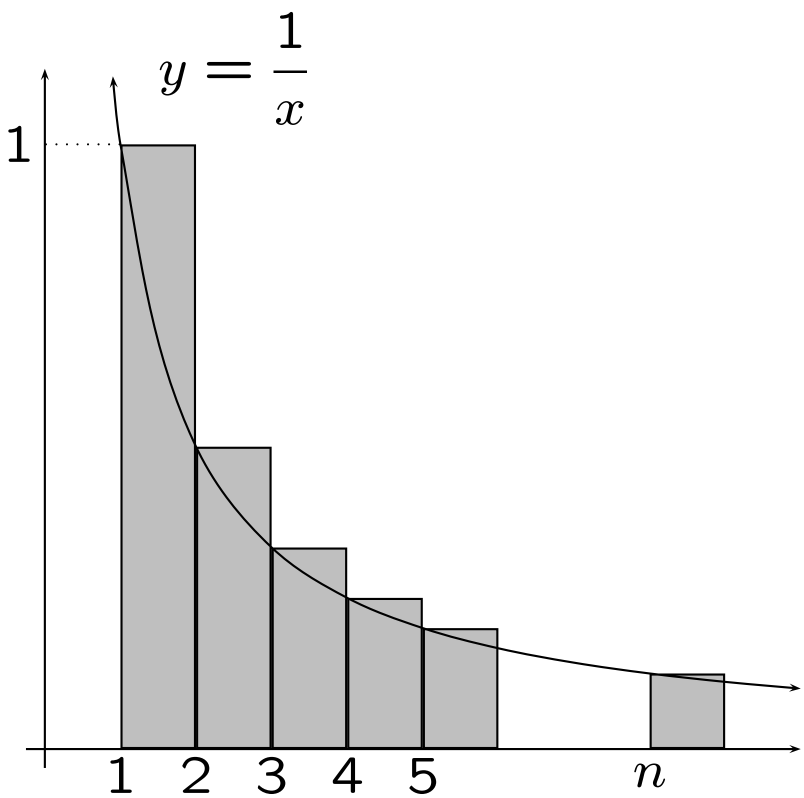
The harmonic series diverges

There are two very common approaches.

Oresme's classical proof...

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n} &= 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)\\ &\qquad +\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)\\ &\qquad\qquad +\left(\frac{1}{9}+\cdots+\frac{1}{16}\right)\cdots\\ &> 1+\frac{1}{2}+\frac{2}{4}+\frac{4}{8}+\frac{8}{16}+\cdots\\ &= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots\end{aligned}$$

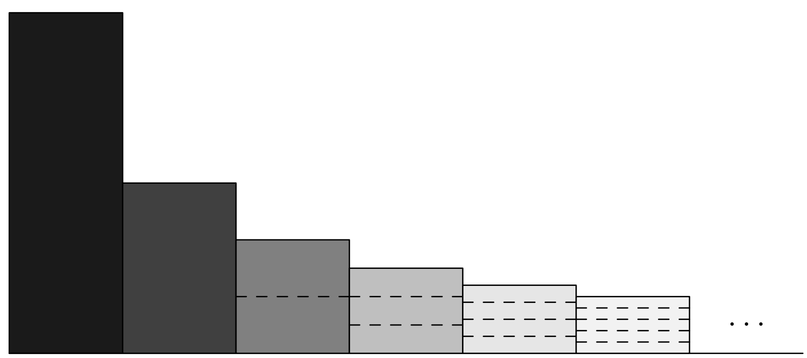
Comparing sums and integrals...

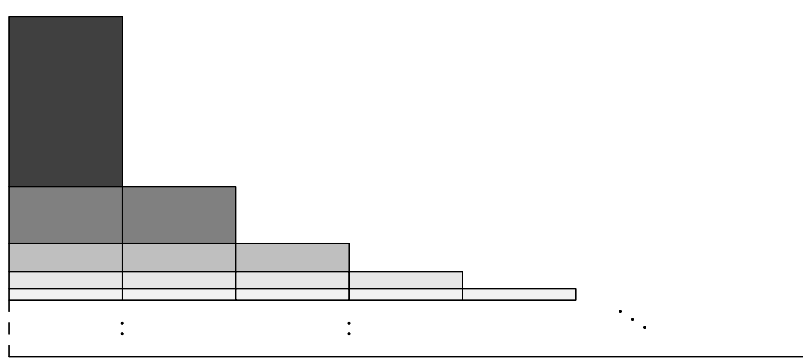


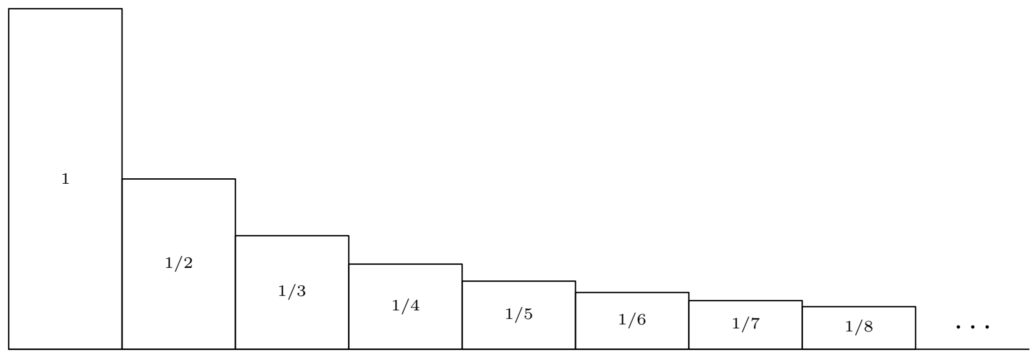
$$\int_1^{n+1} \frac{dx}{x} < 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n}$$

$$\ln(n+1) < \sum_{k=1}^{n} \frac{1}{k}$$

Some more unusual proofs establish divergence by regrouping terms in order to show that the harmonic series is greater than itself.



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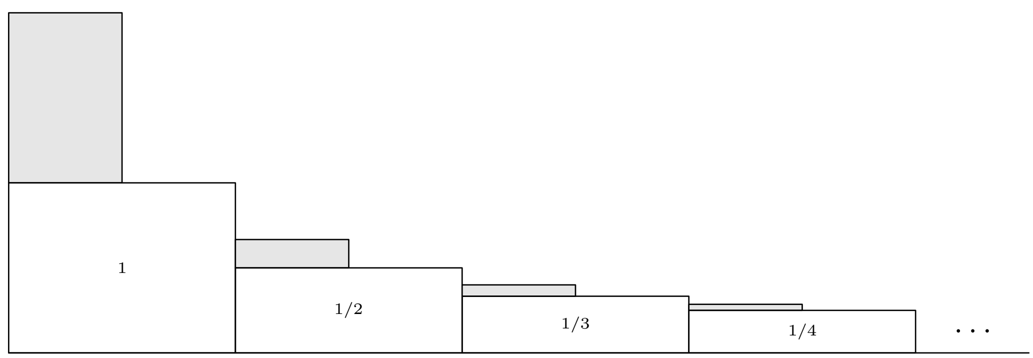
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Euler's Formula

$e^{ix} = \cos x + i \sin x$

This formula is useful when solving 2nd-order, linear, constant-coefficient DEs. Most authors use Taylor series to establish Euler's formula. Here's an easy way:

Both $y = e^{ix}$ and $y = \cos x + i \sin x$ satisfy the following IVP

$$\frac{dy}{dx} = iy, \quad y(0) = 1.$$

Since the IVP must have a unique solution, we're done.

Some integrals using Euler's formula

The integration formulas

$$\int e^x \cos x \, dx = \frac{e^x}{2}(\cos x + \sin x) + C$$

and

$$\int e^x \sin x \, dx = \frac{e^x}{2}(\sin x - \cos x) + C$$

are usually derived by using integration by parts.

Here is a different approach:

$$\int e^x \cos x \, dx \quad + \quad i \int e^x \sin x \, dx$$

$$= \int e^x \, e^{ix} \, dx$$

$$= \int e^{(1+i)x} \, dx$$

$$= \left(\frac{1}{1+i}\right) e^{(1+i)x}$$

$$= \left(\frac{1-i}{2}\right) e^x e^{ix}$$

$$= \frac{e^x}{2}(1-i)(\cos x + i \sin x)$$

$$\frac{e^x}{2}(\cos x + \sin x) \quad + \quad i \, \frac{e^x}{2}(-\cos x + \sin x)$$

The natural log and powers of x

The natural logarithm is often defined by means of a definite integral:

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0.$$

An alternative definition can be useful:

$$\ln x = \lim_{k\to 0} \int_1^x t^{k-1}\,dt = \lim_{k\to 0} \frac{x^k - 1}{k}.$$

- Most of the familiar properties of the logarithm can easily be derived by using the new definition.

- The graph of $y = \ln x$ is the "limit" of the graphs of $y = (x^k - 1)/k$.

- It is clear that $\ln x$ grows slower than any power of x.

- It is easy to shown that

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n.$$

An alternative to integrating by parts

Normally evaluating $\int x^n e^{ax}\,dx$ is an exercise in repeated integration by parts. An alternative approach introduces students to the method of undetermined coefficients.

$$\int x^4 e^{5x}\,dx = e^{5x}(Ax^4+Bx^3+Cx^2+Dx+E)+\text{Const.}$$

Now differentiate and solve for the undetermined coefficients.

$$\begin{array}{ccccccccccc}
5A & & & & & & & & & = & 1 \\
4A & + & 5B & & & & & & & = & 0 \\
 & & 3B & + & 5C & & & & & = & 0 \\
 & & & & 2C & + & 5D & & & = & 0 \\
 & & & & & & D & + & 5E & = & 0
\end{array}$$

This bidiagonal system is easily solved to give

$$A = 1/5, \qquad B = -4/25, \qquad C = 12/125,$$

$$D = -24/625, \qquad E = 24/3125,$$

and therefore

$$\int x^4 e^{5x}\, dx =$$

$$e^{5x}(\frac{1}{5}x^4 - \frac{4}{25}x^3 + \frac{12}{125}x^2 - \frac{24}{625}x + \frac{24}{3125}) + c.$$

Exact DEs and integration by parts

Exact differential equations such as

$$(2xy - 9x^2)dx + (2y + x^2 + 1)dy = 0$$

are often solved by partial integration.

Students who misunderstand the solution method sometimes "solve" the equation as follows:

$$\int (2xy - 9x^2)\,dx + \int (2y + x^2 + 1)\,dy = C$$

$$x^2y - 3x^3 \quad + \quad y^2 + x^2y + y \quad = \quad C,$$

where the integrals are evaluated partially and added.

The method is correct if we compute total integrals, using integration by parts for mixed-variable integrals:

$$(2xy - 9x^2)dx + (2y + x^2 + 1)dy = 0$$

$$\int 2xy\,dx - \int 9x^2\,dx + \int 2y\,dy + \int x^2\,dy + \int 1\,dy = C.$$

The first and fourth integrals require integration by parts, but only one of them must be evaluated.

Letting $u = y$ and $dv = 2x\,dx$, we have

$$\int 2xy\,dx = x^2y - \int x^2\,dy.$$

Substituting this into the equation above, we obtain the correct solution

$$x^2y - \int x^2\,dy - 3x^3 + y^2 + \int x^2\,dy + y = C$$

or

$$x^2y - 3x^3 + y^2 + y = C.$$

Geometric series

There are a number of ways to find the sum of a geometric series. This uncommon approach uses a telescoping series.

$$a - ar + ar - ar^2 + ar^2 + \cdots$$
$$-ar^{n-1} + ar^{n-1} - ar^n = a - ar^n$$

$$a(1-r) + ar(1-r) + ar^2(1-r) + \cdots$$
$$+ar^{n-1}(1-r) = a - ar^n$$

$$\sum_{k=0}^{n-1} ar^k = \frac{a - ar^n}{1 - r}$$

Distance from point to plane

Find the distance from the point $(-1, 5, 3)$ to the plane described by $x + 2y - 2z = 6$.

Solution

A unit vector perpendicular to the plane is given by

$$\vec{n} = \frac{1}{3}(\hat{\imath} + 2\hat{\jmath} - 2\hat{k}).$$

The line through $(-1, 5, 3)$ perpendicular to the plane is described by the following set of parametric equations:

$$x = -1 + t/3, \qquad y = 5 + 2t/3, \qquad z = 3 - 2t/3$$

The parameter t represents the signed distance from the point $(-1, 5, 3)$ to the point (x, y, z).

Now we find t so that (x, y, z) lies on the plane.

$$(-1 + t/3) + 2(5 + 2t/3) - 2(3 - 2t/3) = 6$$

$$3 + 3t = 6$$

$$t = 1$$

The point lies 1 unit from the plane, and the closet point on the plane is $(-2/3, 17/3, 7/3)$.

A sum via integration by parts

The improper integral

$$\int_1^0 \ln x \, dx = 1$$

is easily evaluated by using standard integration by parts (and L'Hôpital's rule).

Tabular integration by parts is an interesting alternative.

signs	u and du/dx	dv/dx and $\int dv$
$+$	$\ln x$	1
$-$	$1/x$	x
$+$	$-1/x^2$	$x^2/2$
$-$	$2/x^3$	$x^3/6$
$+$	$-6/x^4$	$x^4/24$
$\vdots$	$\vdots$	$x^5/120$
$\vdots$	$\vdots$	$\vdots$

From the table, we have

$$\int_1^0 \ln x\,dx = x\ln x - \frac{x}{2} - \frac{x}{6} - \frac{x}{12} - \frac{x}{20} - \cdots \Big|_1^0$$

$$\int_1^0 \ln x \, dx = x \ln x - \frac{x}{2} - \frac{x}{6} - \frac{x}{12} - \frac{x}{20} - \cdots \Big|_1^0$$

After evaluating at $x = 0$ and $x = 1$, we have

$$1 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots.$$

Horner's method

In algebra classes, synthetic division is usually described as trimmed-down or collapsed long division.

Students learn later that synthetic division can be used to evaluate polynomials.

Alternatively, one can show directly how synthetic division is equivalent to Horner's method for evaluating polynomials.

Suppose we wish to evaluate $p(2)$ when

$$p(x) = 3x^3 + 2x^2 - 5x + 2.$$

- Rewrite $p(x) = x(x(x(3) + 2) - 5) + 2$

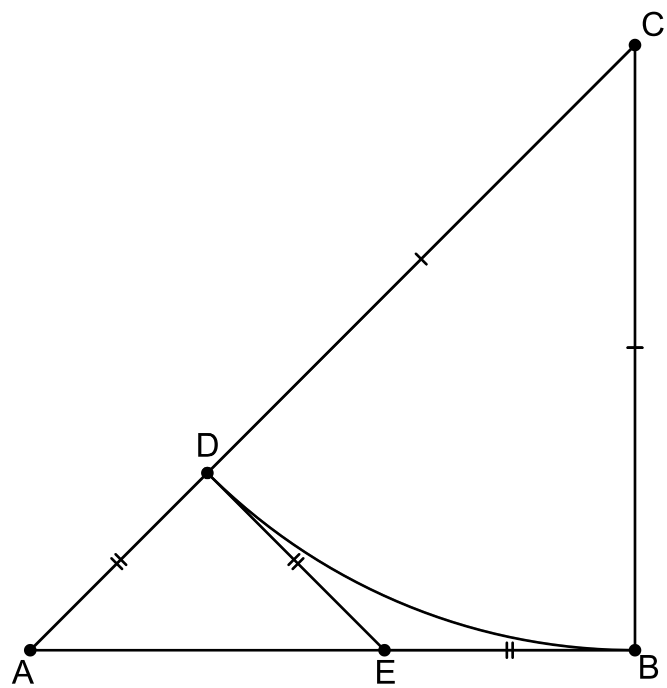
- The evaluation of $p(2)$ corresponds to the synthetic division table below:

	3	2	−5	2
2		6	16	22
	3	8	11	24

- The numbers in the final row of the table come from within the consecutive nested parentheses.

The square root of 2 is irrational

For any isosceles right triangle with integer sides lengths, there exists a smaller one with the same property.



If $\triangle ABC$ is an isosceles right triangle with integer side lengths, then so is $\triangle EDA$.

As a consequence, $\sqrt{2}$ is irrational.

Exponential decay

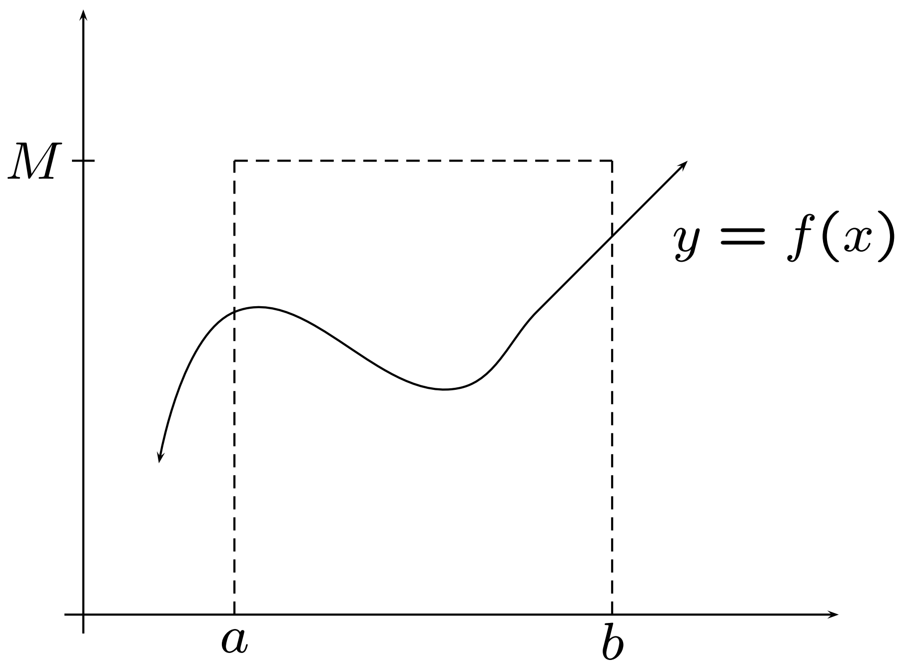
- Most two-year college students don't get to do real radioactive decay experiments.

- Hands-on experience is probably valuable.

- Dice and random number generators provide accessible alternatives.

A large group of dice can be used to simulate decay: throw the dice, remove the sixes, and repeat.

Monte Carlo Integration



- Choose M such that $M \geq f(x)$ on $[a, b]$.

- Generate N random points (x, y), where $x \in [a, b]$ and $y \in [0, M]$.

- $P =$ number of points satisfying $y \leq f(x)$.

- $\int_a^b f(x)dx \approx (P/N) \cdot M(b - a)$.

Monte Carlo integration is especially useful for double and triple integrals.

For example: Find the volume of the solid in the first octant bounded by the graphs of

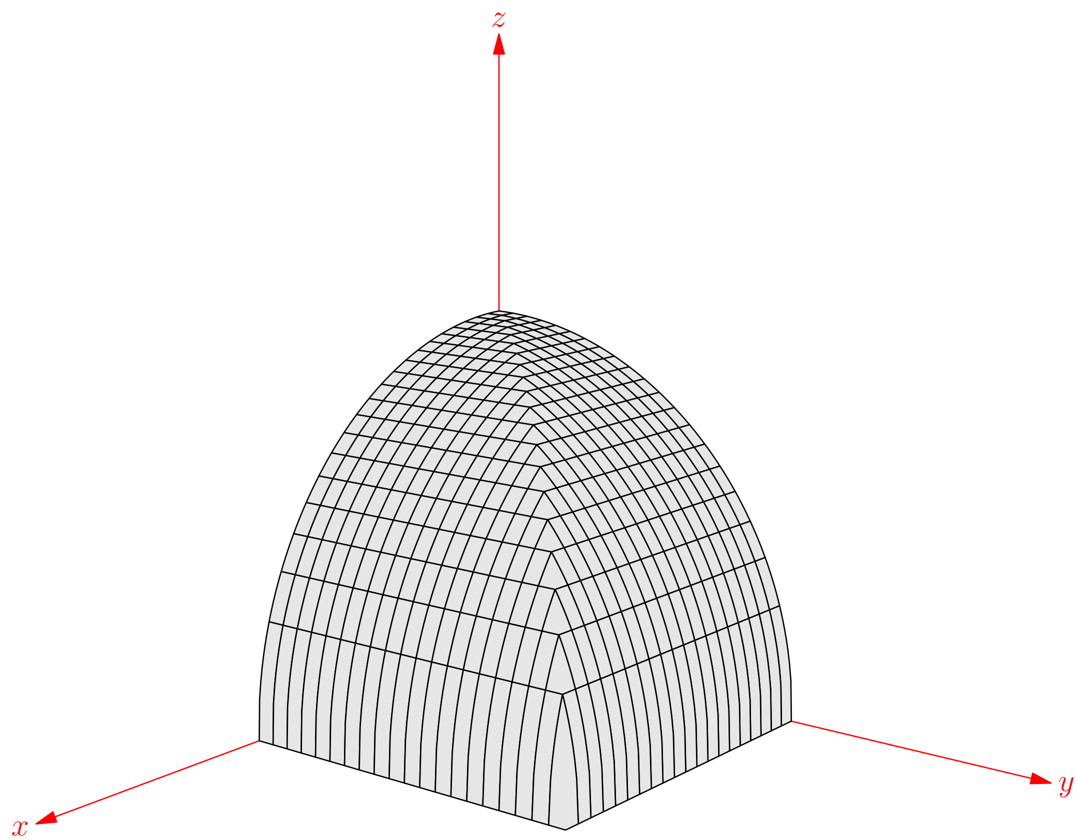
$$x^2 + z^2 = 1$$

and

$$y^2 + z^2 = 1.$$

Our students could describe the region very well, but they had trouble setting up an appropriate iterated integral.

Monte Carlo integration is easy and provides good results.



z
x
y

Another way to sum a series

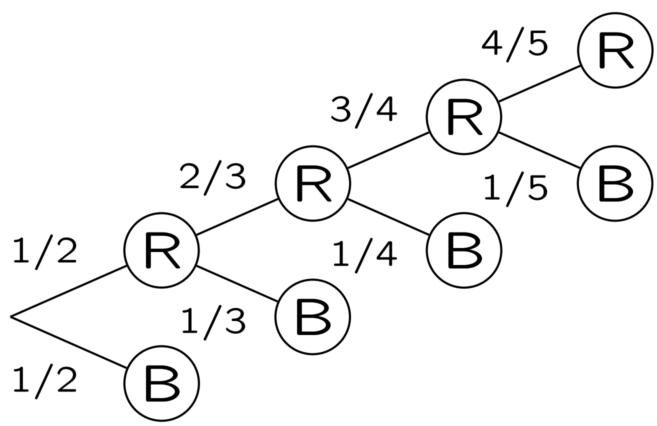
Here is an unusual approach to summing the series

$$\sum_{i=1}^{\infty} \frac{1}{i(i+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots.$$

Consider the following game:

> A red marble and a blue marble are placed into an urn. A marble is selected at random. If the marble is blue, you win. Otherwise, replace the red marble, add another red marble, and repeat the process until you win.

This tree diagram shows the probabilities associated with the first few stages of the game.



- Let B_n be the event of drawing a blue marble (i.e. winning) on the nth draw. The probability of winning (eventually) is given by

$$P(B_1) + P(B_2) + P(B_3) + P(B_4) + \cdots =$$

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots.$$

- Let R_n be the event of drawing n consecutive red marbles, and notice that

$$P(R_n) = \frac{1}{n+1}.$$

- Since $P(R_n) \to 0$, the events B_i exhaust the sample space. So you must eventually draw a blue marble, and the sum above must converge to 1.

It is easy to show that the expected number of draws required to win is given by

$$\sum_{n=2}^{\infty} \frac{1}{n}.$$

Since the harmonic series diverges, this is a game you will win, but it should take forever.

Thanks for attending.

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Slides and handouts are available at
http://skifowit.prairiestate.edu