Assessing Conceptual Understanding in the Calculus Sequence

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If we expect our students to understand concepts, then we must assess conceptual understanding. What does it mean to have a conceptual understanding?

- Students can generalize from particular examples.
- Students can apply and adapt ideas to new situations.
- Students can approach problems visually, numerically, or algebraically, and convert from one representation to another.
- Students can associate meaning with results.
- Students can connect old ideas with new ideas.
- Students can understand the limitations of an idea.

Although there are a number of ways to assess conceptual understanding, the use concept problems is probably the most popular and effective. However, good concept problems can be difficult to write. In her article *What Happened to Tests?* (available at http://www.maa.org/saum/maanotes49/77.html), Ross identifies five question styles that are particularly useful when assessing conceptual understanding.

Creation items. Examples: Give an example of a function that is continuous everywhere, but is not differentiable at $x = 0$ and $x = 1$. Sketch the graph of a function such that...

Reverse questions. Example: If students have been finding slopes of tangent lines, ask them to find a function whose graph has a specified slope at a specified point.

Transition between representation. Example: Given a table of values of a function, sketch a rough graph of the derivative.

Understanding in the presence of technology. Example: Sketch the graph $y = ax^2$ if $a < 0$.

Interpretation. Example: Water is poured at a constant rate into a right circular cylinder. Sketch the graph of the height of the water vs time. Explain your reasoning.

Sample Concept Problems

Concept problems are becoming more and more popular, even in traditional textbooks and classrooms. If we wish to accurately assess conceptual understanding, then our students must be informed, and they must know that their grades will reflect their understanding. Concept problems, however, cannot become routine. By their very nature, concept problems should place students into new situations where they will have to apply and adapt what they have learned.

On the following pages, you will find samples of some of the concept problems that have been used in Calculus I at Prairie State College. Most of these problems have come from or been inspired by problems in popular reform textbooks or mathematics journals.

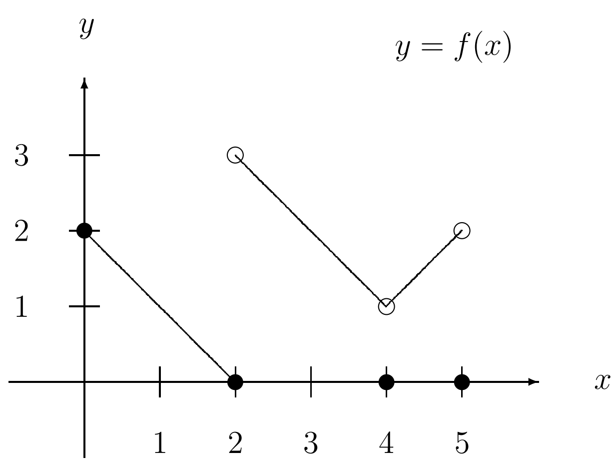
1. Suppose the function f has the following properties:

   - f is defined and continuous for all real numbers
   - f is an odd function
   - $f(2) = 5$

   Compute $\lim_{x \to -2} f(x)$. Carefully explain your reasoning.

2. Suppose that f is an odd function. Does knowing that $\lim_{x \to 0^+} f(x) = 3$ tell you anything about $\lim_{x \to 0^-} f(x)$? About $\lim_{x \to 0} f(x)$? Give reasons for your answers.

3. Consider the function f whose graph is shown below. Determine $f(4)$ and $\lim_{x \to 4} f(x)$.



4. The following table gives the values of the continuous functions f and g at selected points near $x = 0$.

x	-0.01	-0.001	-0.0001	0.0001	0.001	0.01
$f(x)$	4.98	4.997	4.9999	4.9998	4.987	4.96
$g(x)$	0.092	0.0013	0.00002	0.00002	0.0013	0.091

   Given this data, what would be a reasonable approximation for $\lim_{x \to 0} \dfrac{f(x)}{g(x)}$? Explain your reasoning.

5. Each row of the table below gives some information about a function f. Fill in each blank entry with an appropriate word or number. In some cases there may be more than one correct answer.

Continuous at $x = 2$	$f(2)$	$\lim_{x \to 2^-} f(x)$	$\lim_{x \to 2^+} f(x)$
Yes	5	5	
No	7		7
No		-1	-1
	2	2	2
Yes		1	

6. The following table gives the values of the polynomial function P at selected x-values.

x	-2	-1	0	1	2
$P(x)$	3	5	8	0	-5

Evaluate each limit.

(a) $\lim_{x \to -2^+} 3P(x) + 2$

(b) $\lim_{x \to \pi/2} P(\sin x)$

(c) $\lim_{x \to 0} \sqrt[3]{P(x)}$

(d) $\lim_{x \to 1} \dfrac{x^2}{|P(x)|}$

7. Some values of the function f are shown in the table below.

x	-0.1	-0.01	0	0.01	0.1
$f(x)$	9.2	4.1	1	-6.1	11.5

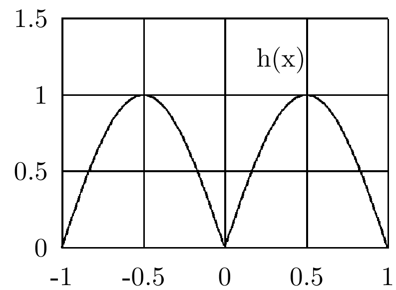
Based only on this information, could it be true that $\lim_{x \to 0} f(x) = 100$? Explain your reasoning.

8. Suppose that f is a differentiable function of x with the following properties:

   - $f(x) > 0$ for all x, and
   - $f'(x) = f(x)$.

   Let $G(x) = f(\frac{1}{3}x^3 - 4x)$ and find intervals on which G is increasing and decreasing.

9. Consider the function $y = h(x)$ whose graph is shown below. Find the critical points of h and determine intervals on which $h'(x)$ is positive and negative. Give reasons for your answers.

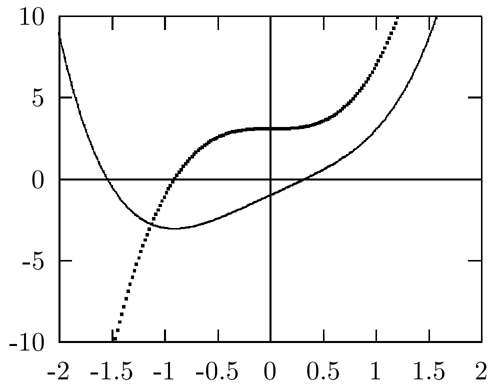


10. Suppose f is continuous and differentiable on $[a, b]$. Show that if $f(a) = f(b)$, then f has a critical point on $[a, b]$.

11. The table below gives the values of a function f and it first two derivatives at selected values of x. Determine which row gives the data for f, which row gives the data for f', and which row gives the data for f''. Explain your reasoning.

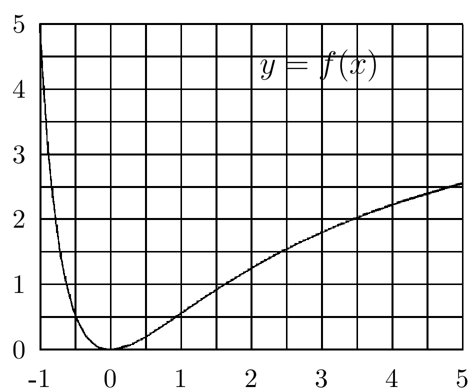
x	0.00	0.33	0.66	1.00	1.33	1.66	2.00	2.33	2.66	3.00
$A(x)$	0.00	0.64	1.14	1.38	1.28	0.84	0.08	−0.89	−1.91	−2.83
$B(x)$	0.00	0.11	0.41	0.84	1.30	1.66	1.82	1.69	1.22	0.42
$C(x)$	2.00	1.78	1.16	0.24	−0.83	−1.85	−2.65	−3.07	−3.00	−2.40

12. The following figure shows the graph of a function and its derivative. Which is which? Give at least two reasons to support your conclusion.



13. At exactly 2pm, David began walking away from his car at a constant rate. After walking 300 meters in five minutes, David stopped for two minutes. He then turned around and ran back to his car at a constant rate. The return trip took David one minute. Sketch the graph of the function $v(t)$ that represents David's velocity t minutes into his trip.

14. The figure below shows the graph of the function $y = f(x)$ for $-1 \leq x \leq 5$.



Approximate the x-value(s) for which

(a) $f'(x) = 0$

(b) $f'(x) < 0$

(c) $f'(x) > 0$

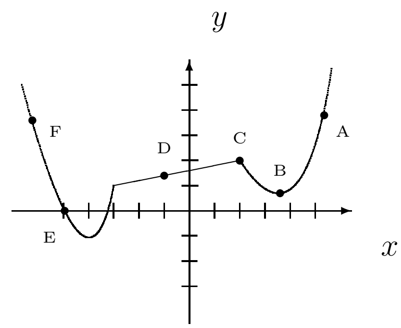
(d) $f''(x) < 0$

(e) $f''(x) > 0$

15. If $f'(5) = 7$, $f(5) = 0$, $g(1) = 5$, and $g'(1) = 3$, determine $\frac{d}{dx} f(g(x))$ when $x = 1$.

16. If $\lim_{x\to\infty} f(x) = 50$ and $f'(x)$ is positive for all x, what is $\lim_{x\to\infty} f'(x)$? (Assume this limit exists.) Explain your answer with a picture.

17. Consider the function f whose graph is shown below.



Referring to the labeled points, find a point at which

(a) $f'(x) = 0$

(b) $0 < f'(x) < 1$

(c) $f'(x) > 1$

(d) $f(x) = 0$

(e) $f'(x) < 0$

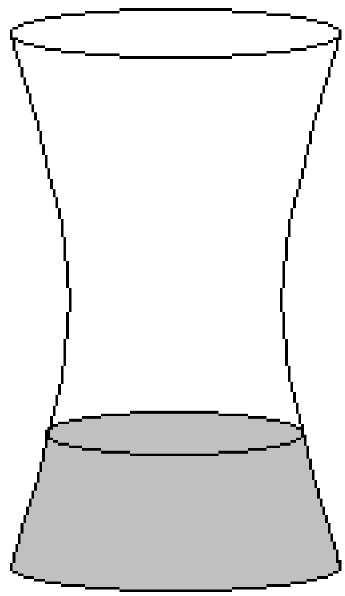
(f) $f'(x)$ is not defined

18. Suppose that both f and g are increasing, differentiable functions. Is the function $f + g$ increasing? Carefully explain your reasoning.

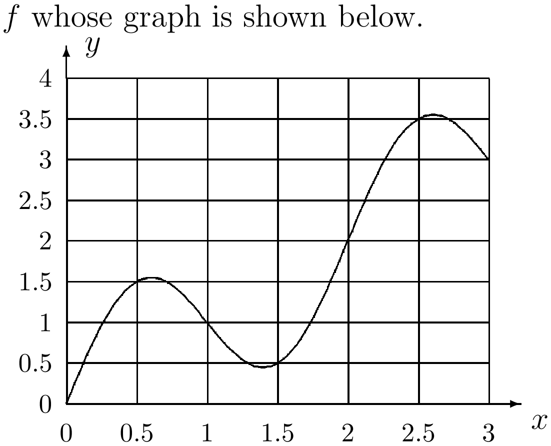
19. The table belows gives the values of the function f at selected points. Find a reasonable approximation for $f'(1)$.

x	0.8	0.9	1.0	1.1	1.2
$f(x)$	1.67	1.85	2.03	2.21	2.38

20. Suppose that water is being poured into the vase shown below at a constant rate measured in cubic centimeters per second. Sketch the graph of the function $y = h(t)$ that gives the height of the water in centimeters after t seconds. Explain the concavity of your graph.



21. Consider the function f whose graph is shown below.



(a) Find approximate intervals along the x-axis on which

i. $f'(x) < 0$

ii. $f'(x) > 0$

iii. $f''(x) < 0$

iv. $f''(x) > 0$

(b) Approximate the x-value(s) for which

i. $f'(x) = 0$

ii. $f''(x) = 0$

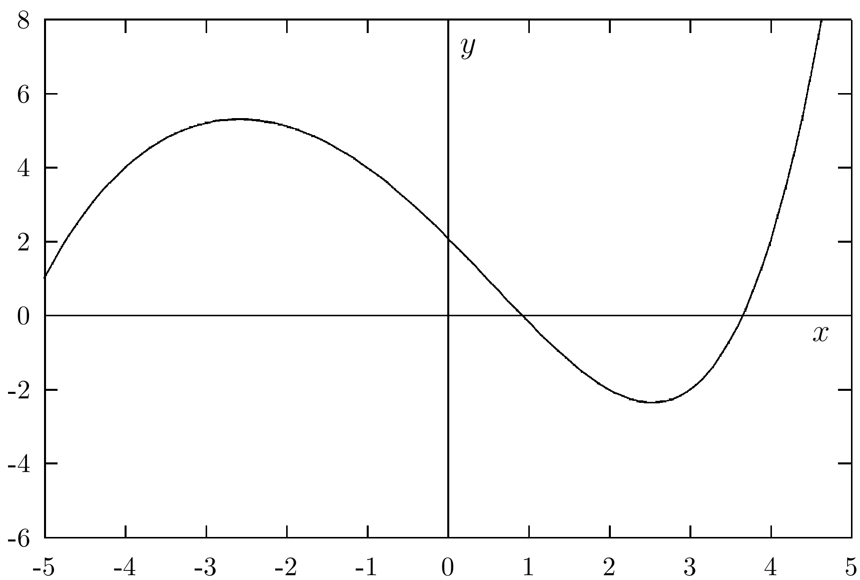
iii. $f(x)$ is a relative maximum

22. A Calculus I student was asked to compute the slope of the line tangent to the graph of $y = 2^x$ at the point where $x = 0$. Since she didn't know a formula for finding dy/dx, she decided to approximate the slope by using her calculator. She determined some values of the expression $(2^h - 1)/h$ for h-values close to zero.

h	$(2^h - 1)/h$
10^{-2}	0.695555
10^{-4}	0.693171
10^{-6}	0.693147
10^{-8}	0.693147
10^{-10}	0.693148
10^{-12}	0.693223
10^{-14}	0.688388
10^{-16}	0
10^{-18}	0

Explain what the student was doing. What value should she use for the approximate slope of the tangent line?

23. The graph of the function f is shown below. For each part of this problem, tell which number is greater and why.



(a) $f'(-4)$ or $f'(-1)$

(b) $f'(1)$ or $f(1)$

(c) $f''(-3)$ or $f''(3)$

(d) $f'(-3)$ or $f'(4)$

(e) $-f''(-2.5)$ or $f''(2.5)$

24. Given a function f and its first and second derivatives, f' and f'', which of these would you use and how would you use it to determine

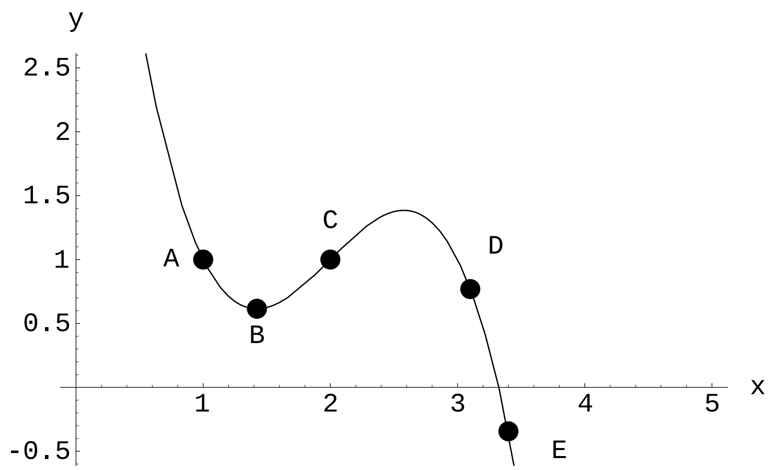
(a) if f has a critical point at $x = 3$?

(b) the zeros of f?

(c) if the graph of f has an inflection point at $x = -1$?

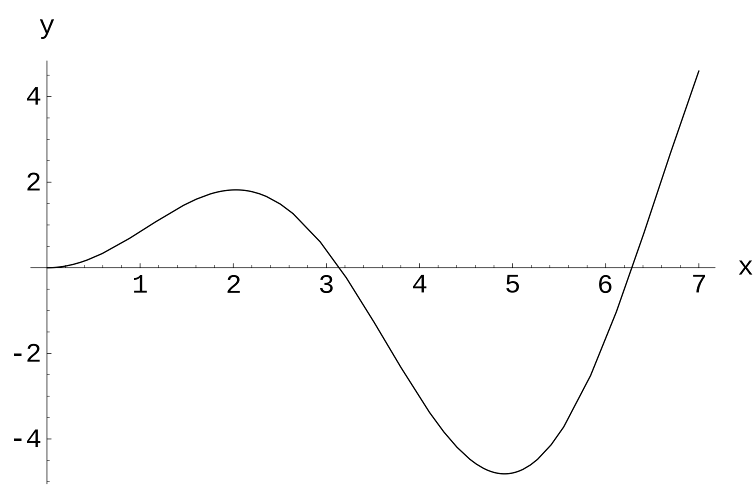
(d) intervals on which f is decreasing?

25. At which labeled point(s) might it be possible that $\frac{dy}{dx} = \frac{d^2y}{dx^2}$. Explain.



26. The graph of the function f is shown below. Referring to this graph, arrange the following quantities in ascending order.

$$f'(1), \quad f'(5), \quad f(5), \quad f(7), \quad f'(3)$$

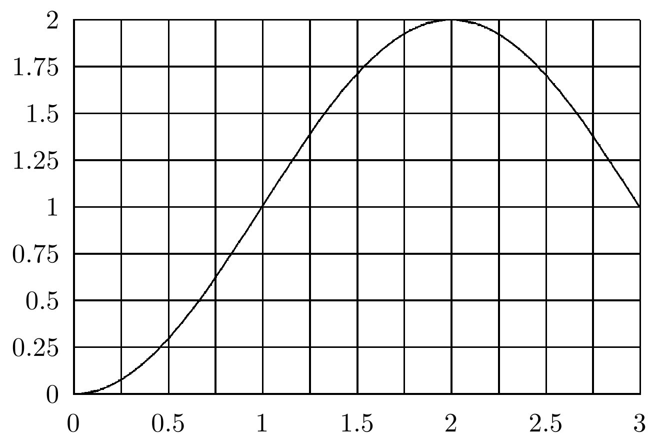


27. Suppose the function f satisfies the following conditions:

- f is differentiable on $[0, 5]$
- $f'(x) < 2$ for all x in $[0, 5]$
- $f(0) = 0$

Use the Mean Value Theorem to show that $f(5) < 10$.

28. A particle is moving along a straight line with velocity $v(t)$, $0 \leq t \leq 3$. The graph of v is shown below.



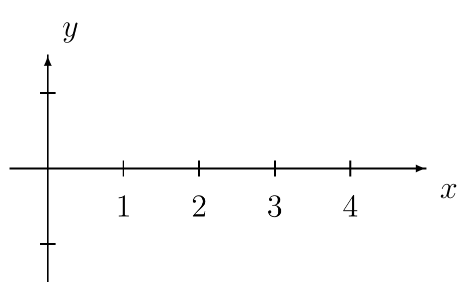
(a) Give a reasonable approximation for the particle's acceleration at $t = 1$.

(b) Over what time interval is the particle decelerating (slowing down)?

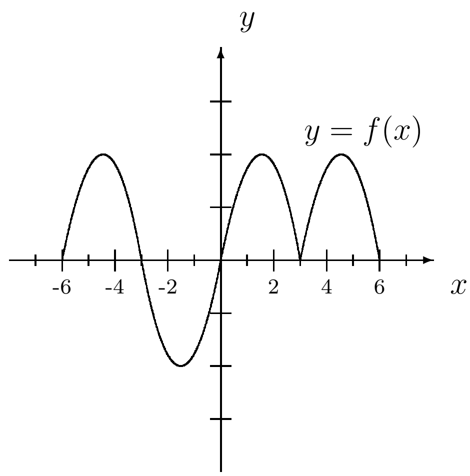
(c) Give a reasonable estimate for the particle's displacement over the first second.

29. Sketch the graph of a continuous function f such that

$$\int_0^2 f(x)\,dx > 0 \qquad \text{and} \qquad \int_0^4 f(x)\,dx < 0.$$



30. Refer to the figure below. In each case, find a value of k that makes the expression true.



(a) $\int_{-6}^{6} f(x)\,dx = \int_{-6}^{k} f(x)\,dx + \int_{k}^{6} f(x)\,dx$

(b) $\int_{-3}^{3} f(x)\,dx = k$

(c) $\int_{0}^{6} f(x)\,dx = \int_{0}^{3} kf(x)\,dx$

(d) $\int_{0}^{3} f(x)\,dx + \int_{6}^{k} f(x)\,dx = 0$

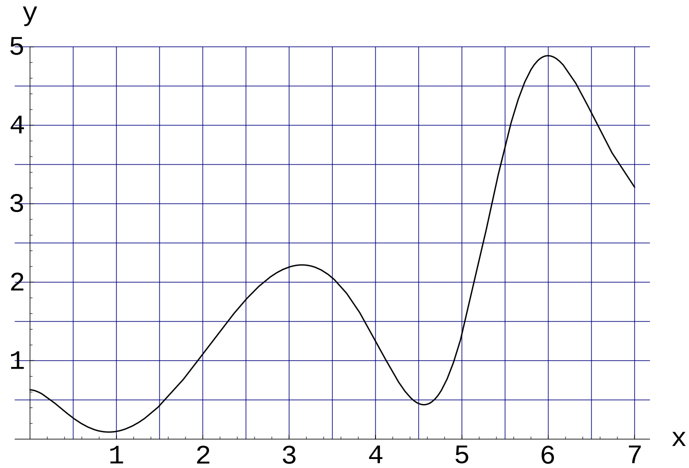
31. Jason determined that the average value of the function $h(x) = \sqrt{2 + 5\tan^2 x}$ on $[0, \pi/4]$ is -5.32561. Explain why his result cannot be correct.

32. The area under the curve $y = f(x)$ from $x = 2$ to $x = 4$ is approximated by a Riemann sum of the form

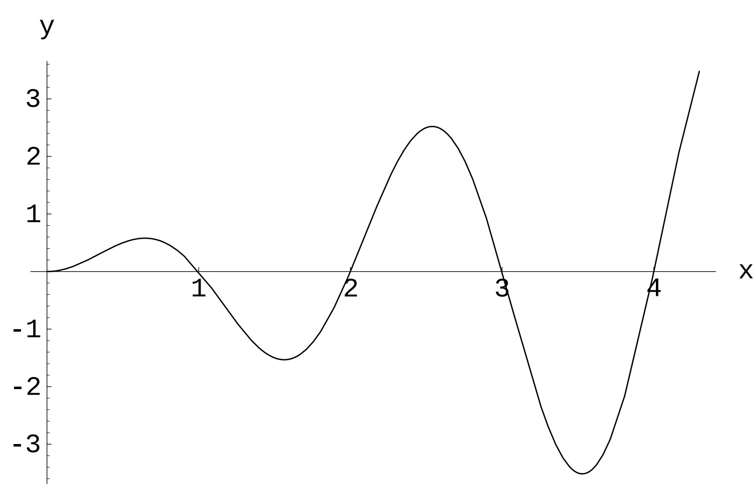
$$\sum_{k=1}^{n} (x_k^2 + x_k)\Delta x,$$

where the interval [2,4] is divided into n subintervals of equal width Δx and x_k is some point in the k^{th} subinterval. What is the exact area of the region?

33. The graph of the function $y = f(x)$ is shown below. Use the grid squares to approximate the value of the definite integral $\int_0^7 f(x)\,dx$. Explain your reasoning. (Notice that the grid squares are 0.5 units by 0.5 units.)



34. The graph of the function $y = f(x)$ is shown below.

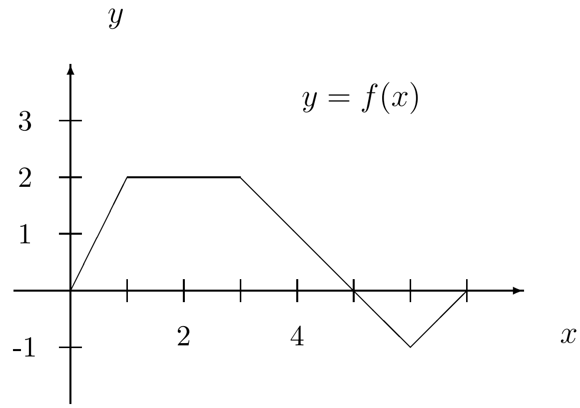


Let A, B, C, and D be real numbers such that

$$A = \int_0^1 f(x)\,dx \qquad B = \int_0^2 f(x)\,dx \qquad C = \int_0^3 f(x)\,dx \qquad D = \int_0^4 f(x)\,dx$$

Arrange A, B, C, and D in ascending order. Explain your reasoning.

35. Given the graph of the function f, determine each of the following.



(a) $\displaystyle\int_1^5 f(x)\,dx$

(b) $f'(5)$

(c) $\displaystyle\int_2^1 f(x)\,dx$

(d) $f''(2)$

(e) $\displaystyle\int_5^7 f(x)\,dx$

36. Consider the limit

$$\lim_{\|P\|\to 0} \sum_{k=1}^{N} (\cos^2 \pi c_k)\Delta x_k,$$

where P is a partition of [0,2]. Explain how we know that this limit exists.

37. The table below gives the values of a function f and the function $g(x) = \int_0^x f(t)\,dt$ at selected values of x. Which row gives the data for f and which row gives the data for g? Give at least two reasons for your answer.

x	0.00	0.50	1.00	1.50	2.00	2.50	3.00	3.50	4.00
$A(x)$	0.00	0.24	0.84	1.50	1.82	1.50	0.42	-1.23	-3.03
$B(x)$	0.00	0.04	0.30	0.89	1.74	2.60	3.11	2.93	1.86

38. Suppose you know that $\displaystyle\int_1^3 3x^2\,dx = R$ and $\displaystyle\int_1^3 2x\,dx = S$. What is the value of $\displaystyle\int_3^1 (x^2 - x)\,dx$ in terms of R and S?