The Method of Gnomons and a New Scheme for Approximating Square Roots*

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1 Introduction

There are a number of methods for approximating square roots with rational numbers. One remarkably simple technique was presented in the 1998 NCTM yearbook (Mason, 1998). This technique is probably best described as a pictorial approach to linear interpolation. Interestingly, it can be used as a starting point for a new method of approximating square roots to any desired accuracy. In addition, the technique has some interesting generalizations.

The idea described in the yearbook article is based on the classical Greek notion that perfect squares, or square numbers, can be depicted by square arrangements of dots (see Figure 1). The principle square root of a square number is

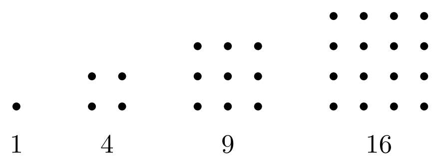

Figure 1: The first four square numbers

simply the number of dots along a side of the corresponding square. Of course this does not apply to non-square numbers, but it does lead to a quick and easy method for approximating square roots. This method is illustrated in Figure 2.

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$\sqrt{11} \approx 3\frac{2}{7}$ $\qquad$ $\sqrt{8} \approx 2\frac{4}{5}$

Figure 2: Approximating square roots

In applying this technique, squares are constructed by adjoining dots arranged in the shape of a backward L. These arrangements of dots are called *gnomons*. A gnomon is a shape which, when added to a figure, yields another figure similar to the original. It is not difficult to determine a bound on the error made when this method of gnomons is used to approximate a square root.

2 Error and Consequences

The error bound

Suppose that x is some positive integer between the integers n^2 and $(n+1)^2$. If we attempt to arrange x dots into the shape of a square, we will obtain an $n \times n$ square with $x - n^2$ dots left over. Since n^2 and $(n+1)^2$ differ by $2n+1$, our approximation for $\sqrt{x}$ will be given by

$$K(n,x) = n + \frac{x - n^2}{2n+1}.$$

For the sake of illustration, Table 1 gives several approximations, including those from Figure 2, and their corresponding errors.

Until now, we have been restricting our attention to integral values of x.

x	n	$K(n,x)$	$\sqrt{x} - K(n,x)$
8	2	$2\frac{4}{5}$	0.02843
11	3	$3\frac{2}{7}$	0.03091
21	4	$4\frac{5}{9}$	0.02702
48	6	$6\frac{12}{13}$	0.00513
109	10	$10\frac{9}{21}$	0.01173

Table 1: Several square root approximations

This helps to motivate the technique (see Figure 2), but the restriction is not necessary. For any fixed nonnegative integer n, K is a linear function of the real variable x whose graph passes through the points (n^2, n) and $\big((n+1)^2, n+1\big)$. At this point, the proof of the following theorem is a straight-forward optimization problem, much like those encountered in a beginning calculus course.

Theorem 1 *Let n be a nonnegative integer. If $n^2 \leq x \leq (n+1)^2$, then*

$$|\sqrt{x} - K(n,x)| = \sqrt{x} - K(n,x) \leq \frac{1}{4+8n}.$$

Equality holds if and only if $x = (n + \frac{1}{2})^2$.

Although this result was described in the *College Mathematics Journal* over twenty-five years ago (McKenna, 1976), one very interesting consequence seems to have never been discussed: the error made in approximating $\sqrt{100x}$ is significantly less than that made in approximating $\sqrt{x}$. Since $\sqrt{x}$ can be obtained from $\sqrt{100x}$ by simply shifting the position of the decimal point, a more accurate approximation can be obtained with little or no extra cost.

Refining the approximations

Suppose we wish to approximate $\sqrt{17}$. Since 17 is between 4^2 and 5^2, we find that $\sqrt{17} \approx K(4,17) = 4\frac{1}{9} \approx 4.11111$ and, according to Theorem 1,

$$\text{Error } = \sqrt{17} - K(4,17) \leq \frac{1}{36} \approx 0.03.$$

However, we could approximate $\sqrt{1700}$ with greater accuracy, and thereby obtain a better approximation for $\sqrt{17}$. Specifically, since 1700 is between 41^2 and 42^2, we find that

$$\sqrt{1700} \approx K(41, 1700) = 41\frac{19}{83} \approx 41.2289$$

so that

$$\sqrt{17} \approx \frac{K(41,1700)}{10} = \left(\frac{1}{10}\right)\left(41\frac{19}{83}\right) \approx 4.12289$$

It follows from Theorem 1 that

$$\frac{\sqrt{1700}}{10} - \frac{K(41,1700)}{10} \leq \frac{\frac{1}{332}}{10}.$$

Therefore the error in our new approximation is less than $1/3320 \approx 0.0003$.

Why stop here? We could approximate $\sqrt{170000}$ or $\sqrt{17000000}$ to obtain even better results. But of course, there is a catch. Without an accurate approximation for a square root, how do we obtain the consecutive perfect squares that bound the radicand? In other words, how do we find the n in the $K(n,x)$? Fortunately, we can use our technique recursively, finding new n's from old n's. The following result is a simple consequence of the inequality in Theorem 1, and it provides exactly what we need.

Corollary 1 *Let $\lfloor u \rfloor$ represent the greatest integer less than or equal to the real number u and let n be a positive integer. If $n^2 \leq x \leq (n+1)^2$, then*

$$\lfloor 10K(n,x) \rfloor^2 \leq 100x \leq \left(\lfloor 10K(n,x) \rfloor + 2 \right)^2 .$$

The algorithm

In the case of $\sqrt{17}$ and $\sqrt{1700}$, Corollary 1 assures us that 1700 is between 41^2 and 43^2. We could simply compute 42^2 to determine the required interval. If we continue with this example, our next step would be to approximate $\sqrt{170000}$. According to Corollary 1, 170000 is between 412^2 and 414^2, and we would compute 413^2 to refine our interval. When all is said and done, we find that

$$\sqrt{170000} \approx K(412, 170000) = 412\frac{256}{825} \approx 412.310303$$

so that

$$\sqrt{17} \approx \frac{K(412, 170000)}{100} = \left(\frac{1}{100}\right)\left(412\frac{256}{825}\right) \approx 4.12310303.$$

Theorem 1 guarantees that the error in this approximation is less than $1/330000 \approx 0.000003$.

In general, given a real number $x \in [1, 100)$ and a positive integer n such that $x \in [n^2, (n+1)^2]$, the following algorithm proceeds through M iterations of our method, returning an approximation for $\sqrt{x}$.

(1) $n_1 = n$

(2) **for** $i = 1, 2, \ldots, M-1$

(3) $\quad x_i = K(n_i, 10^{2i-2}x)$ $\qquad$ $(\, x_i \approx 10^{i-1}\sqrt{x}\,)$

(4) $\quad r_i = x_i / 10^{i-1}$ $\qquad$ $(\, r_i \approx \sqrt{x}\,)$

(5) $\quad n_{i+1} = \lfloor 10 x_i \rfloor$

(6) $\quad$ **if** $(n_{i+1} + 1)^2 < 100^i x$ $\qquad$ *Based on Cor. 1, lines 5–8*

(7) $\qquad n_{i+1} = n_{i+1} + 1$ $\qquad$ *find the perfect squares that*

(8) $\quad$ **end if** $\qquad$ *bound the next radicand*

(9) **next** i

(10) $r_M = K(n_M, 10^{2M-2}x)/10^{M-1}$ $\qquad$ *The final approximation*

(11) **return** r_M

As a consequence of Theorem 1, we can find an upper bound for the error at each step of the algorithm.

Corollary 2 *Suppose $x \in [1, 100)$ and n is a positive integer such that $x \in [n^2, (n+1)^2]$. Referring to the algorithm above,*

$$0 \leq \sqrt{x} - r_i \leq \left(\frac{1}{4 + 8n_i}\right)\left(\frac{1}{10^{i-1}}\right) \leq \frac{1}{8n \cdot 10^{2i-2}},$$

for $i = 1, 2, \ldots, M$.

Table 2 summarizes the results of the our algorithm when it is applied to $x = 2$. The error bounds shown in the table are those given by Corollary 2.

i	n_i	r_i	Error Bound	$\sqrt{x} - r_i$
1	1	$\frac{4}{3}$	1.25×10^{-1}	8.09×10^{-2}
2	14	$\frac{41}{29}$	1.25×10^{-3}	4.20×10^{-4}
3	141	$\frac{20011}{14150}$	1.25×10^{-5}	8.62×10^{-6}
4	1414	$\frac{400081}{282900}$	1.25×10^{-7}	5.94×10^{-8}
5	14142	$\frac{200005153}{141425000}$	1.25×10^{-9}	4.14×10^{-10}
6	141421	$\frac{20000020331}{14142150000}$	1.25×10^{-11}	8.11×10^{-12}

Table 2: Algorithm results when $x = 2$

Remarks

Although Corollaries 1 and 2 depend on x being greater than or equal to 1, the algorithm's restriction of x to the interval $[1, 100)$ is merely a matter of convenience. In such a case, a moment's consideration will give the perfect squares that bound x. Any other positive x could be written in the form $x = r \times 10^{2k}$ where $r \in [1, 100)$, and then the algorithm could be applied to r. For example, if $x = 5739.3 = 57.393 \times 10^2$, then $r = 57.393$ and r is between the perfect squares 49 and 64.

We see from Corollary 2 that the number of correct digits in an approximation increases by about two with each iteration. (For a specific example, see the $\sqrt{x} - r_i$ column in the table above.) While the algorithm moves along quickly, it is far slower than Newton's method or other higher order methods. Nonetheless, the new algorithm does have its selling points. First, it is very easily motivated. Its derivation is straight-forward, and it is easily illustrated. The algorithm is also remarkably elementary; the proof of its convergence requires no advanced mathematics, and its usage involves no difficult computations. It is quite simple to perform two or three iterations by hand, after which you can be certain of the accuracy you have obtained. In addition, knowledge of the algorithm improves one's number sense when it comes to square roots.

Notice that lines 6–8 of the algorithm apply the check step in which the bounding interval is refined. In practice, it turns out that this step is not necessary and may be skipped with little effect on the quality of the results.

3 Generalizing the Method of Gnomons

Although the method of gnomons was not introduced with quadratic equations in mind, we can obviously think of it as a way of generating an approximation for the positive solution of $x^2 = A$, where A is a positive integer. Taking this point of view, the method can easily be generalized.

The square numbers are only one group of a class of numbers called *figurate*. A figurate number is a number that can be depicted by an arrangement of dots in the shape of a common geometric figure (often a regular polygon). For example, the first four triangular numbers are shown in Figure 3.

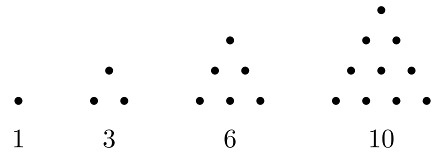

Figure 3: The first four triangular numbers

The nth triangular number is $n(n+1)/2$. Therefore, a method of gnonoms can be used to approximate the unique positive solution of

$$\frac{1}{2}x^2 + \frac{1}{2}x = A,$$

where A is a positive integer. If we apply the technique when $A = 13$, we find that the quadratic equation $\frac{1}{2}x^2 + \frac{1}{2}x = 13$ has a positive solution that is approximately $4\frac{3}{5}$. This is illustrated in Figure 4.

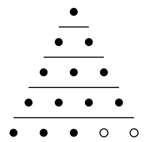

Figure 4: $\frac{1}{2}x^2 + \frac{1}{2}x = 13 \Rightarrow x \approx 4\frac{3}{5}$

The method also generalizes quite nicely to rectangular numbers of the form $n^2 + n$, $n^2 + 2n$, etc. and to other figurate numbers. In each case, one could establish a result analogous to Theorem 1. For example, the following generalization of Theorem 1 can be illustrated using rectangular arrays of dots where the length is k more than the width.

Theorem 2 *Let n and k be positive integers, and suppose A is a real number in the interval $[n(n+k), (n+1)(n+1+k)]$. If R is the unique positive solution of $x(x+k) = A$, then*

$$R \approx n + \frac{A - n(n+k)}{2n+1+k}$$

and

$$0 \leq R - \left(n + \frac{A - n(n+k)}{2n+1+k}\right) \leq \frac{1}{4k+4+8n}.$$

Equality holds if and only if $x = \frac{1}{4}(1 + 2k + 4n + 4kn + 4n^2)$.

References

Mason, D. E. (1998). Capsule lessons in alternative algorithms for the classroom. In Morrow, L. J. and Kenney, M. J., editors, *The Teaching and Learning of Algorithms in School Mathematics*, pages 91–98, Reston, VA. The National Council of Teachers of Mathematics, Inc.

McKenna, J. E. (1976). Interpolation and square roots. *College Mathematics Journal*, 7(4):49–50.

Appendix

Proof of Theorem 1

Let n be a fixed nonnegative integer and define f for $x \in [n^2, (n+1)^2]$ by

$$f(x) = \sqrt{x} - K(n, x) = \sqrt{x} - n - \frac{x - n^2}{2n+1}.$$

f is continuous on $[n^2, (n+1)^2]$ and differentiable on $(n^2, (n+1)^2)$, and

$$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{2n+1}.$$

The only critical point of f occurs where $f'(x) = 0$. This is the point where $x = n^2 + n + \frac{1}{4} = (n + \frac{1}{2})^2$ and $f(x) = \frac{1}{4+8n}$.

Now $f''(x) = -x^{-3/2}/4$ and therefore the graph of f is concave down on $(n^2, (n+1)^2)$. Furthermore $f(n^2) = f((n+1)^2) = 0$. So f is nonnegative on $[n^2, (n+1)^2]$ and attains a unique maximum at the point $\left((n + \frac{1}{2})^2, \frac{1}{4+8n}\right)$.

Proof of Corollary 1

Let $\lfloor u \rfloor$ represent the greatest integer less than or equal to the real number u, let $\lceil u \rceil$ represent the least integer greater than or equal to the real number u, and let n be a positive integer. If $x \in [n^2, (n+1)^2]$, then, by Theorem 1,

$$K(n, x) \le \sqrt{x} \le K(n, x) + \frac{1}{4+8n}.$$

It follows that

$$10K(n, x) \le 10\sqrt{x} \le 10K(n, x) + \frac{10}{4+8n} < 10K(n, x) + 1$$

so that

$$\lfloor 10K(n, x) \rfloor \le 10\sqrt{x} < \lceil 10K(n, x) + 1 \rceil.$$

Now since $\lceil u + 1 \rceil = \lceil u \rceil + 1$, we have

$$\lfloor 10K(n, x) \rfloor \le 10\sqrt{x} < \lceil 10K(n, x) \rceil + 1$$

and upon squaring, we get

$$\lfloor 10K(n, x) \rfloor^2 \le 100x < \left(\lceil 10K(n, x) \rceil + 1\right)^2.$$

Finally, since $\lceil u \rceil + 1 \le \lfloor u \rfloor + 2$, we obtain the result

$$\lfloor 10K(n, x) \rfloor^2 \le 100x < \left(\lfloor 10K(n, x) \rfloor + 2\right)^2.$$

Proof of Corollary 2

Suppose that n is a positive integer and $x \in [n^2, (n+1)^2]$. Referring to the algorithm, we first establish that

$$n_i^2 \leq 10^{2i-2}x \leq (n_i + 1)^2$$

for $i = 1, 2, \ldots, M$. The proof is by induction on i.

By definition of n_1 ($n_1 = n$), we have

$$n_1^2 \leq x \leq (n_1 + 1)^2.$$

Now suppose the inequality above holds for $i = k$. Then

$$n_k^2 \leq 10^{2k-2}x \leq (n_k + 1)^2$$

and, by Corollary 1, we have

$$\lfloor 10K(n_k, 10^{2k-2}x) \rfloor^2 \leq 100 \cdot 10^{2k-2}x \leq \left(\lfloor 10K(n_k, 10^{2k-2}x) \rfloor + 2 \right)^2$$

or

$$\lfloor 10K(n_k, 10^{2k-2}x) \rfloor^2 \leq 10^{2k}x \leq \left(\lfloor 10K(n_k, 10^{2k-2}x) \rfloor + 2 \right)^2 .$$

Since $n_{k+1} = \lfloor 10K(n_k, 10^{2k-2}x) \rfloor$ (lines 3 and 5 of the algorithm), we now have

$$n_{k+1}^2 \leq 10^{2k}x \leq (n_{k+1} + 2)^2 .$$

After evaluating $(n_{k+1} + 1)^2$, comparing its value against $10^{2k}x = 100^k x$, and renaming n_{k+1} if necessary (lines 6–8 of the algorithm), we get

$$n_{k+1}^2 \leq 10^{2k}x \leq (n_{k+1} + 1)^2 .$$

It follows that

$$n_i^2 \leq 10^{2i-2}x \leq (n_i + 1)^2 \tag{1}$$

for $i = 1, 2, \ldots, M$, and the first part of the proof is finished.

Now, each x_i is defined by $x_i = K(n_i, 10^{2i-2}x)$, and since the inequality (1) holds, it follows from Theorem 1 that

$$\sqrt{10^{2i-2}x} - x_i \leq \frac{1}{4 + 8n_i}.$$

Upon dividing by 10^{i-1}, we get

$$\sqrt{x} - \frac{x_i}{10^{i-1}} \leq \left(\frac{1}{4 + 8n_i} \right) \left(\frac{1}{10^{i-1}} \right).$$

Since $r_i = \dfrac{x_i}{10^{i-1}}$ (line 4 of the algorithm), we see that

$$\sqrt{x} - r_i \le \left(\frac{1}{4 + 8n_i}\right)\left(\frac{1}{10^{i-1}}\right). \tag{2}$$

Finally, by taking square roots in inequality (1), we have

$$n_{i-1} \le 10^{i-2}\sqrt{x} \le n_{i-1} + 1, \quad i = 2, 3, 4, \ldots$$

and

$$n_i \le 10^{i-1}\sqrt{x} \le n_i + 1, \quad i = 1, 2, 3, \ldots. \tag{3}$$

After multiplying the first inequality by 10, we have

$$10n_{i-1} \le 10^{i-1}\sqrt{x} \le 10(n_{i-1} + 1). \tag{4}$$

Notice that by inequality (3), n_i and $n_i + 1$ are consecutive positive integers bounding $10^{i-1}\sqrt{x}$. From inequality (4), the positive (non-consecutive) integers $10n_{i-1}$ and $10(n_{i-1}+1)$ also bound $10^{i-1}\sqrt{x}$. Since there are no integers between n_i and $n_i + 1$, we must have

$$10n_{i-1} \le n_i < n_i + 1 \le 10(n_{i-1} + 1).$$

Therefore

$$10n_{i-1} \le n_i \le 10^{i-1}\sqrt{x} \le n_i + 1 \le 10(n_{i-1} + 1).$$

Specifically, $10n_{i-1} \le n_i$ and inductively, $10^{i-1}n_1 \le n_i$.

So at last we have

$$\underbrace{\sqrt{x} - r_i \le \left(\frac{1}{4 + 8n_i}\right)\left(\frac{1}{10^{i-1}}\right)}_{\text{Inequality (2)}} < \underbrace{\left(\frac{1}{8n_i}\right)\left(\frac{1}{10^{i-1}}\right) \le \left(\frac{1}{8n_1}\right)\left(\frac{1}{10^{2i-2}}\right)}_{\text{since } 10^{i-1}n_1 \le n_i}.$$

Proof of Theorem 2

Let n and k be positive integers, and suppose A is a real number in the interval $[n(n+k), (n+1)(n+1+k)]$. The unique positive solution, R, of $x(x+k) = A$ is given by the quadratic formula:

$$R = \frac{-k + \sqrt{k^2 + 4A}}{2}.$$

Now define f as a function of A by

$$f(A) = \frac{-k + \sqrt{k^2 + 4A}}{2} - \left(n + \frac{A - n(n+k)}{2n + 1 + k}\right).$$

Notice that $f(A)$ is the error made in approximating the unique positive solution of $x(x+k) = A$ by $n + \frac{A-n(n+k)}{2n+1+k}$.

f is a differentiable function of A on $(n(n+k), (n+1)(n+1+k))$ and

$$f'(A) = \frac{1}{\sqrt{k^2+4A}} - \frac{1}{2n+1+k}.$$

The only critical point of f occurs where $f'(A) = 0$. This is the point where $A = \frac{1}{4}(1+2k+4n+4kn+4n^2)$ and $f(x) = \frac{1}{4+4k+8n}$.

Now $f''(A) = -2(k^2+4A)^{-3/2}$ and therefore the graph of f is concave down on $(n(n+k), (n+1)(n+1+k))$. Furthermore

$$f(n(n+k)) = f((n+1)(n+1+k)) = 0.$$

So f is nonnegative on $[n(n+k), (n+1)(n+1+k)]$ and attains a unique maximum at the point $\left(\frac{1}{4}(1+2k+4n+4kn+4n^2), \frac{1}{4+4k+8n}\right)$.

Mathematica Code

The Mathematica code for the algorithm is given here.

```
K[n_,x_]:= n + (x-n^2) / (2n+1)


SQR[x_,n_,M_]:=Module[{xx,nn,rr},
   nn=n;
   Do[
      xx=K[nn,10^(2i-2) x];
      rr=10^(1-i) xx;
      nn=Floor[10 xx];
      If[(nn+1)^2 < 100^i x, nn=nn+1], {i,1,M-1}];
   rr=K[nn,10^(2M-2) x] 10^(1-M);
   Print[rr]]
```

To produce the results in Table 2, enter `SQR[2,1,1]`, `SQR[2,1,2]`, etc.