Serious About the Harmonic Series

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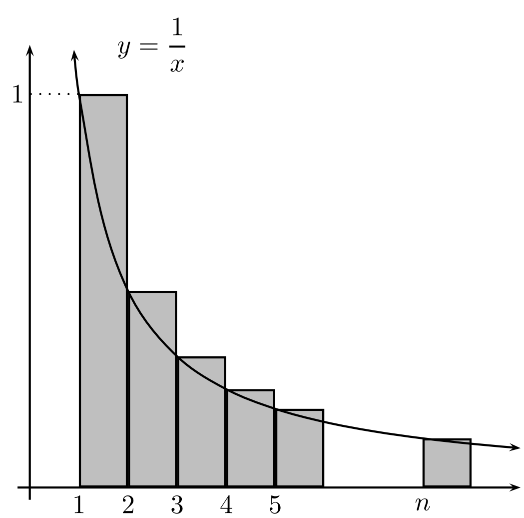
With its rich and diverse history, applications, and divergence proofs, the harmonic series provides the instructor with a wealth of opportunities. The presenters will describe how they have taken advantage of these opportunities to engage calculus students. The presentation will focus mostly on unusual proofs and applications.

1 Notation

- Harmonic Series: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$
- nth partial sum of the harmonic series: $\displaystyle H_n = \sum_{k=1}^{n} \frac{1}{k}$

2 The harmonic series diverges



$$\int_1^{n+1} \frac{dx}{x} = \ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n} = H_n$$

3 The harmonic series diverges very slowly

The harmonic series diverges, but it does so incredibly slowly. For example, the sum of the first 13,000 terms barely exceeds 10. How many terms would be required to reach 1000? Using the lower bound on H_n that is given above, we are sure to have $H_n > 1000$ if we have $\ln(n+1) > 1000$. In order for this inequality to be satisfied, n must be nearly 10^{435}. To get a good idea of just how many terms this is, consider the following:

The world's most powerful supercomputer can do about 70 trillion operations per second. The amount of time required to compute the sum of the first 10^{435} terms would be

$$(10^{435}\,\text{ops})\left(\frac{1\,\text{sec}}{70\times 10^{12}\,\text{ops}}\right)\left(\frac{1\,\text{hr}}{3600\,\text{sec}}\right)\left(\frac{1\,\text{day}}{24\,\text{hr}}\right)\left(\frac{1\,\text{year}}{365\,\text{days}}\right)\approx 4.5\times 10^{413}\,\text{years}.$$

It is difficult to appreciate the magnitude of this number. Perhaps it will suffice to compare it with the estimated age of the universe—a mere 1.5×10^{10} years.

4 H_n is almost never an integer

Given that the sequence of H_n's diverges, and its does so very slowly, it is rather surprising that, with the exception of $n = 1$, H_n is never an integer. Here is a sketch of the proof:

Consider H_n, $n > 1$, and choose k so that $2^k \le n < 2^{k+1}$. We have

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k} + \cdots + \frac{1}{n}.$$

Now let M be the LCM of all the denominators except 2^k. That is,

$$M = \text{LCM}(1, 2, 3, \ldots, 2^k - 1, 2^k + 1, \ldots, n).$$

A crucial point here is that M has a factor 2^{k-1} but not 2^k.

Multiply H_n and M to get

$$\begin{aligned} M \cdot H_n &= M + \frac{M}{2} + \frac{M}{3} + \cdots + \frac{M}{2^k} + \cdots + \frac{M}{n} \\ &= \text{integer} + \frac{M}{2^k} + \text{integer}. \end{aligned}$$

Based on our definition of M, $M/2^k$ cannot be an integer. Therefore $M \cdot H_n$ cannot be an integer, and it follows that H_n is not an integer.

5 Gabriel's wedding cake

Gabriel's horn is obtained by rotating the graph of $y = 1/x$, $1 \le x < \infty$, about the x-axis. This paradoxical solid has finite volume but infinite surface area. It is sometimes said that the horn can be filled with paint, but cannot be painted.

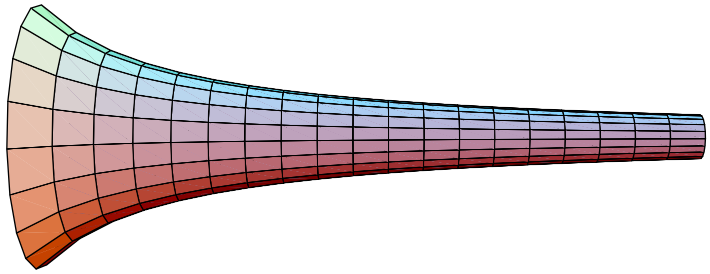


Figure 1: Gabriel's horn

In [4], Fleron describes *Gabriel's wedding cake*, a discrete analogue of Gabriel's horn. Let f be the following piecewise-defined function:

$$f(x) = \begin{cases} 1, & 1 \le x < 2 \\ 1/2, & 2 \le x < 3 \\ & \dots \\ 1/n, & n \le x < n+1 \\ & \dots \end{cases}$$

Now rotate the graph of $y = f(x)$, $1 \le x < \infty$, about the x-axis.

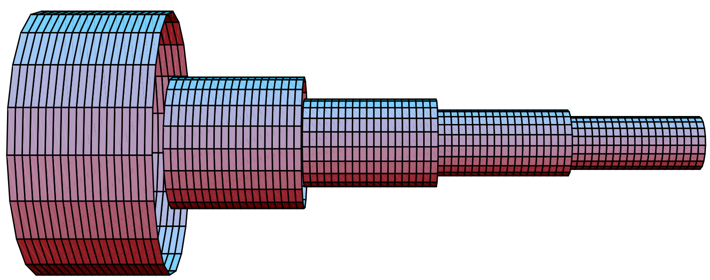


Figure 2: Gabriel's wedding cake

Gabriel's wedding cake has volume given by

$$V = \sum_{n=1}^{\infty} \pi \left(\frac{1}{n}\right)^2 (1) = \pi \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^3}{6}$$

and lateral surface area given by

$$A = \sum_{n=1}^{\infty} 2\pi \left(\frac{1}{n}\right) (1) = 2\pi \sum_{n=1}^{\infty} \frac{1}{n}.$$

Since the harmonic series diverges, Gabriel's wedding cake is a cake you can eat, but cannot frost.

6 The harmonic series diverges

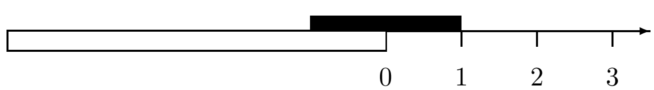
Suppose the harmonic series converges with sum S.

$$\begin{aligned} S &= 1+\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\right)+\left(\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}\right) \\ &\quad +\left(\frac{1}{11}+\cdots+\frac{1}{15}\right)+\left(\frac{1}{16}+\cdots+\frac{1}{21}\right)+\cdots \\ &> 1+\frac{2}{3}+\frac{3}{6}+\frac{4}{10}+\frac{5}{15}+\frac{6}{21}+\cdots \\ &= \frac{2}{2}+\frac{2}{3}+\frac{2}{4}+\frac{2}{5}+\frac{2}{6}+\frac{2}{7}+\cdots \\ &= 2\sum_{n=2}^{\infty}\frac{1}{n} \\ &= 2(S-1). \end{aligned}$$

The inequality $S > 2(S-1)$ implies $S < 2$. Since the fourth partial sum of the harmonic series already exceeds 2, we have an obvious contradiction.

7 The leaning tower of lire

How far can a stack of equal-sized blocks be made to extend from the edge of a table? To answer this, suppose we have an unlimited supply of rectangular blocks, each having length two units and mass one unit. If we make a stack of only one block, then our stack can extend at most 1 unit off the table. In this case, the center of mass of the stack will be at the edge of the table. Let's set up a number line along the table so that the origin is at the edge, and the positive side of the number line extends off the table.



Now lift the one-block stack straight up and place it on top of a single block whose end is at the origin. The new two-block stack is made up of a one-block stack with center at -1 and a one-block stack with center at the origin. Its center of mass is then given by

$$\frac{(-1)(1)+(0)(1)}{2} = -\frac{1}{2}.$$

Therefore the new two-block stack can be pushed 1/2 of a unit off the table. After doing so, the new stack will have center of mass at the origin, and its edge will extend

$$1+\frac{1}{2}$$

units off the table.

We could continue adding blocks one at a time, but instead let's consider the general case. Suppose we have an $(n-1)$-block stack with center of mass at the origin. We lift the stack straight up and place it on top of a single block whose end is at the origin. The new n-block stack is made up of a one-block stack with center of mass at -1 and an $(n-1)$-block stack with center of mass at the origin. The center of mass of the n-block stack is then given by

$$\frac{(-1)(1)+(0)(n-1)}{n} = -\frac{1}{n}.$$

Therefore, the n-block stack can be pushed $1/n$ units off the table so that its center of mass is at the origin.

If we constucted our n-block stack, one block at a time, by lifting and pushing as outlined above, its edge will extend

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$$

units off the table. With enough blocks, we can make our stack extend as far as we'd like!
(The earliest reference to this problem, that we know of, is [5].)

8 The harmonic series diverges

Proposition: For any natural number k, $\dfrac{1}{k}+\dfrac{1}{k+1}+\cdots+\dfrac{1}{3k} > 1.$
Proof:

$$\begin{aligned}
\exp\left(\frac{1}{k}+\frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{3k}\right) &= e^{1/k}\cdot e^{1/(k+1)}\cdot e^{1/(k+2)}\cdots e^{1/(3k)} \\
&> \left(1+\frac{1}{k}\right)\cdot\left(1+\frac{1}{k+1}\right)\cdot\left(1+\frac{1}{k+2}\right)\cdots\left(1+\frac{1}{3k}\right) \\
&= \left(\frac{k+1}{k}\right)\cdot\left(\frac{k+2}{k+1}\right)\cdot\left(\frac{k+3}{k+2}\right)\cdots\left(\frac{3k+1}{3k}\right) \\
&= \frac{3k+1}{k} > 3.
\end{aligned}$$

Corollary: The harmonic series diverges.

9 H_n and record breaking

How often should Chicagoans expect record snowfall in January? Assuming that the amount of snowfall in January of one year has no effect on the amount of snowfall in January of any subsequent year, we have the following.

- The first year of record keeping is a record year.
- The probability that the second year is a record year is $\frac{1}{2}$. So, the expected number of record snowfalls in 2 years is $1+\frac{1}{2}$.
- The probability that the third year is a record year is $\frac{1}{3}$. So, the expected number of record snowfalls in 3 years is $1+\frac{1}{2}+\frac{1}{3}$.
- In general, after n years of observation, we should expect H_n record years.

The following data were collected from the Illinois State Climatologist Office. When all is said and done, record breaking snowfall in January is pretty predictable.

Inches of Snowfall for January, 1960–2004 Measured at O'Hare Airport—Chicago, IL (R denotes a record year)					
Year	Inches	Year	Inches	Year	Inches
1960	3.5 R	1975	3.5	1990	3.2
1961	3.0	1976	10.0	1991	11.1
1962	18.6 R	1977	7.2	1992	5.6
1963	16.8	1978	21.9	1993	15.2
1964	1.6	1979	34.3 R	1994	14.2
1965	11.7	1980	6.2	1995	13.1
1966	15.5	1981	2.0	1996	5.9
1967	25.1 R	1982	22.9	1997	no data
1968	10.4	1983	5.0	1998	no data
1969	3.7	1984	17.2	1999	29.6
1970	9.5	1985	18.9	2000	13.6
1971	10.0	1986	6.9	2001	1.5
1972	7.6	1987	17.3	2002	15.5
1973	0.5	1988	5.4	2003	4.3
1974	7.4	1989	0.4	2004	14.6

Table 1: Chicago snowfall data obtained from the Illinois State Climatologist Office

The following table shows the numbers of Illinois tornadoes for the years 1956–2004. During the 49 years of observation, there were 5 record years. Since $H_{49} \approx 4.5$, perhaps Illinois should not expect a record number of tornadoes any time soon.

Number of Illinois Tornadoes, 1956–2004 (R denotes a record year)					
Year	Tornadoes	Year	Tornadoes	Year	Tornadoes
1956	28 R	1973	63 R	1990	50
1957	42 R	1974	107 R	1991	32
1958	27	1975	46	1992	23
1959	37	1976	27	1993	34
1960	40	1977	33	1994	20
1961	34	1978	13	1995	76
1962	13	1979	12	1996	62
1963	11	1980	14	1997	29
1964	7	1981	33	1998	99
1965	28	1982	35	1999	64
1966	11	1983	14	2000	55
1967	40	1984	34	2001	21
1968	8	1985	15	2002	35
1969	10	1986	22	2003	120 R
1970	17	1987	22	2004	80
1971	16	1988	20		
1972	30	1989	15		

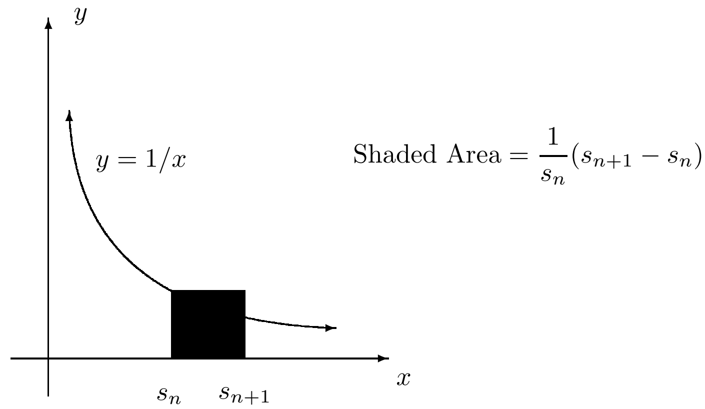
Table 2: Illinois tornado data obtained from The Disaster Center.

10 The harmonic series diverges

For any divergent series, there is a much smaller divergent series. (See [1].)

Proposition: If $a_n > 0$, $\sum_{n=1}^{\infty} a_n$ diverges, and $s_n = a_1 + a_2 + \cdots + a_n$, then $\sum_{n=1}^{\infty} \frac{a_{n+1}}{s_n}$ diverges.

Proof: First notice that $\sum_{n=1}^{\infty} \frac{a_{n+1}}{s_n} = \sum_{n=1}^{\infty} \frac{s_{n+1} - s_n}{s_n}$.



$$\frac{1}{s_n}(s_{n+1} - s_n) > \int_{s_n}^{s_{n+1}} \frac{1}{x}\, dx$$

$$\sum_{n=1}^{\infty} \frac{1}{s_n}(s_{n+1} - s_n) > \int_{s_1}^{\infty} \frac{1}{x}\, dx = \infty$$

Corollary 1: The harmonic series diverges.

Corollary 2: $\sum_{n=1}^{\infty} \frac{1}{nH_n}$ diverges (very, very slowly).

11 The collector's problem

You have just purchased your 10th box of Sugary Goodness breakfast cereal desperately trying to collect all six toys for your child. How many more should you expect to purchase before your set of six toys is complete?

Assuming that each cereal box contains exactly one toy and that each toy is equally likely, we have the following:

- The probability of getting one toy with the first box purchased is 1.

- Given that you have one toy, the probability of getting a second (non-duplicate) toy with your next purchase is 5/6. So, the expected number of boxes you would need to purchase is 6/5.
- Given that you have two distinct toys, the probability of getting a third (non-duplicate) toy with your next purchase is 4/6. So, the expected number of boxes you would need to purchase is 6/4.
- If we continue with this resoning, you should expect to have a complete set after

  $$1+\frac{6}{5}+\frac{6}{4}+\frac{6}{3}+\frac{6}{2}+\frac{6}{1}=6\cdot\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\right)=6\cdot H_6$$

  purchases.

In general, the expected number of purchases necessary to obtain one complete set of n objects is

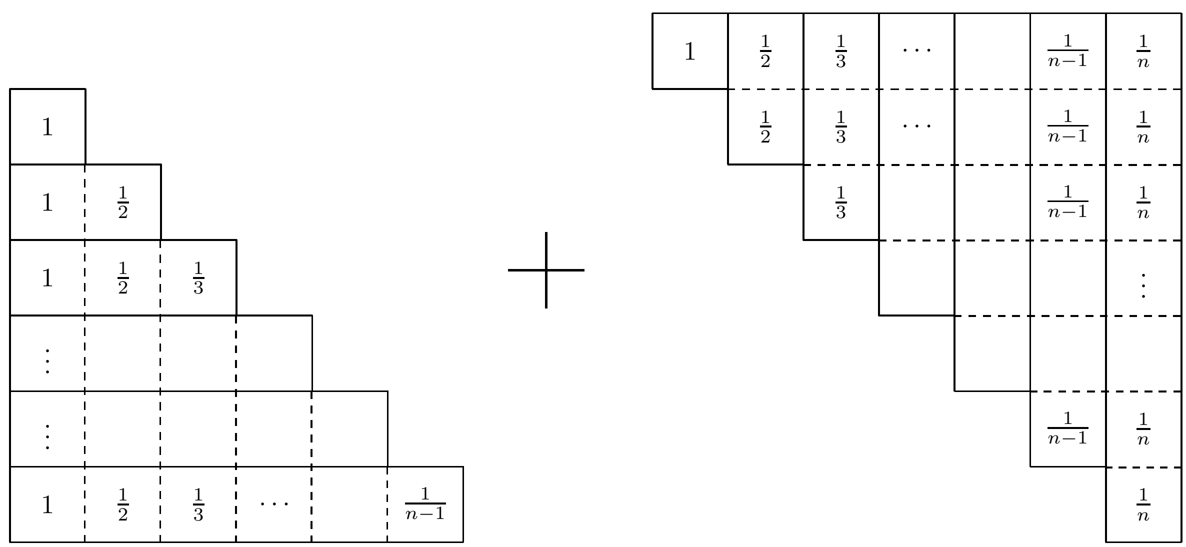
$$n\cdot\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)=nH_n.$$

(For more information on the collector's problem, see [7].)

12 Sums of partial sums

This *Proof Without Words* appears in [6].

$$\sum_{k=1}^{n-1} H_k \;+\; n = nH_n$$



$\sum_{k=1}^{n-1} H_k$ + n

Fit the shapes together for n groups of H_n.

13 The harmonic series diverges

Jacob Bernoulli gave credit for this proof to his brother Johann. An enjoyable account of the history of the proof can be found in the works of Dunham [2, 3].

Consider the series

$$\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right).$$

As it is written on the right, the series is telescoping and converges to 1. With this series serving as an illustration, note that

$$\sum_{n=k}^{\infty}\frac{1}{n(n+1)}=\sum_{n=k}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=\frac{1}{k},\quad k=1,2,3,\ldots.$$

Now suppose that the harmonic series converges with sum S. Then

$$\begin{aligned}
S &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\cdots \\
&= 1+\frac{1}{2}+\frac{2}{6}+\frac{3}{12}+\frac{4}{20}+\frac{5}{30}+\frac{6}{42}+\frac{7}{56}+\cdots \\
&= 1+\left(\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\cdots\right)+\left(\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\cdots\right) \\
&\quad +\left(\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\cdots\right)+\cdots \\
&= 1+\sum_{n=1}^{\infty}\frac{1}{n(n+1)}+\sum_{n=2}^{\infty}\frac{1}{n(n+1)}+\sum_{n=3}^{\infty}\frac{1}{n(n+1)}+\cdots \\
&= 1+1+\frac{1}{2}+\frac{1}{3}+\cdots \\
&= 1+S.
\end{aligned}$$

The contradiction $S=1+S$ concludes the proof.

14 Sums of partial sums

$$\sum_{n=1}^{\infty}\frac{H_{n+1}}{n(n+1)}=\sum_{n=1}^{\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n+1}\right)\frac{1}{n(n+1)}=2$$

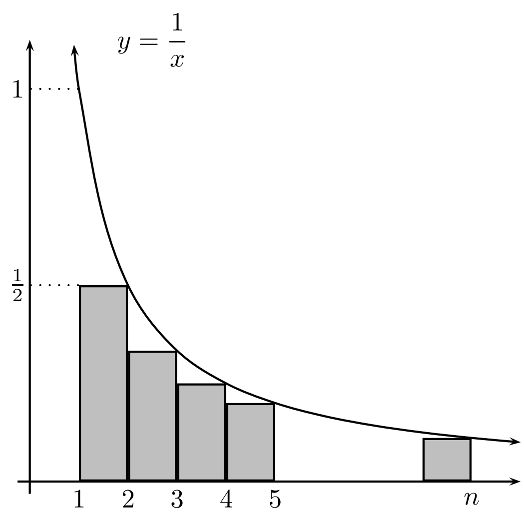
Proof: First notice that

$$\sum_{n=k}^{\infty}\frac{1}{n(n+1)}=\sum_{n=k}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right)=\frac{1}{k}.$$

With this in mind,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_{n+1}}{n(n+1)} &= \frac{1+\frac{1}{2}}{2} + \frac{1+\frac{1}{2}+\frac{1}{3}}{6} + \frac{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}{12} + \frac{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}}{20} + \cdots \\
&= \left(\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots\right) + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots\right) \\
&\quad + \frac{1}{3}\left(\frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots\right) + \frac{1}{4}\left(\frac{1}{12} + \frac{1}{20} + \cdots\right) + \cdots \\
&= \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{k=1}^{\infty} \frac{1}{k+1} \sum_{n=k}^{\infty} \frac{1}{n(n+1)} \\
&= \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \\
&= 1 + 1.
\end{aligned}$$

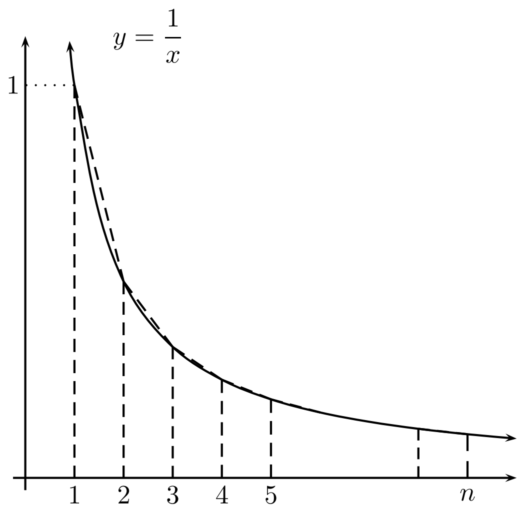
15 Upper bound on H_n



$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} < \int_1^n \frac{1}{x}\, dx$$

$$H_n < 1 + \ln n$$

16 Lower bound on H_n from trapezoid rule



$$\int_1^n \frac{1}{x}\,dx < \frac{1}{2}\left(1+\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}\right) + \cdots + \frac{1}{2}\left(\frac{1}{n-1}+\frac{1}{n}\right)$$

$$\ln n < \frac{1}{2} + \left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}\right) + \frac{1}{2n}$$

$$\ln n - \frac{1}{2} - \frac{1}{2n} < \frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}$$

$$\ln n + \frac{1}{2} + \frac{1}{2n} < 1 + \frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}+\frac{1}{n}$$

$$\ln n + \frac{1}{2} + \frac{1}{2n} < H_n$$

17 Some miscellaneous facts

- Better bounds on H_n: $\ln n - \ln 2 + \frac{5}{4} + \frac{1}{2n} < H_n < \ln n - \ln 2 + \frac{3}{2}, \quad n = 3, 4, 5, \ldots$
- Except for $n = 1$, $n = 2$, and $n = 6$, H_n is not a terminating decimal.
- Wolstenholme's theorem: If $p > 3$ and p is prime, the numerator of H_{p-1} is divisible by p^2.
- $\lim_{n\to\infty}(H_n - \ln n) = \gamma \approx 0.5772$
- For each natural number n, there exists c_n between 0 and 1 such that

$$H_n = \frac{1}{2}\ln(n^2+n) + \gamma + \frac{c_n}{6n^2+6n}.$$

- $-\dfrac{\ln(1-x)}{1-x} = \sum_{n=1}^{\infty} H_n x^n, \quad -1 < x < 1$
- $\sum_{n=1}^{\infty} \dfrac{H_n}{(n+1)2^{n+1}} = \dfrac{(\ln 2)^2}{2}$
- $\sum_{n=1}^{\infty} \dfrac{H_n}{n2^n} = \dfrac{\pi^2}{12}$

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