Serious About the Harmonic Series

Steve Kifowit and Terra Stamps
Prairie State College
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With its rich and diverse history, applications, and divergence proofs, the harmonic series provides the instructor with a wealth of opportunities. The presenters will describe how they have taken advantage of these opportunities to engage calculus students. The presentation will focus mostly on unusual proofs and applications.

Notation

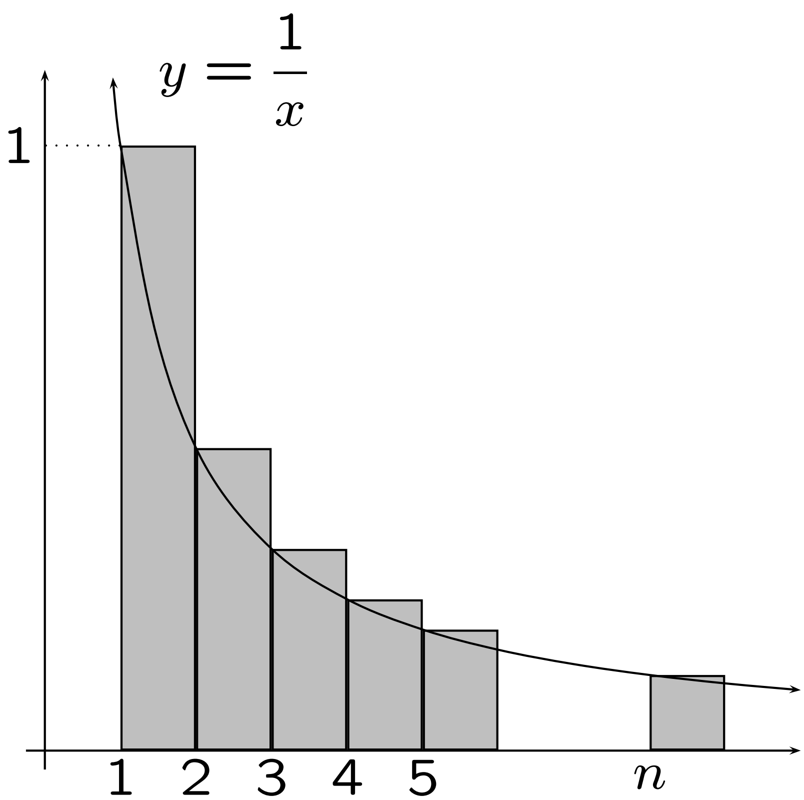
- Harmonic Series:

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

- nth partial sum of the harmonic series:

$$H_n = \sum_{k=1}^{n} \frac{1}{k}$$

The harmonic series diverges



$$\int_1^{n+1} \frac{dx}{x} < 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n}$$

$$\ln(n+1) < H_n$$

Student explorations

- How does the result change if right endpoints are used instead of left endpoints?
- How does the result change if trapezoids are used instead of rectangles?
- Why are results like this important?

The harmonic series diverges very slowly

How many terms of the harmonic series are required before H_n exceeds 1000?

In order for the inequality

$$1000 \leq \ln(n+1) < H_n$$

to be satisfied, n must be approximately 10^{435}.

To sum the first 10^{435} terms of the harmonic series, the most powerful supercomputer would require 4.5×10^{413} years.

To fully appreciate the magnitude of this number, compare it to the estimated age of the universe—a mere 1.5×10^{10} years.

Student explorations

- How many more terms would be required before H_n exceeds 1050? How much more time would be required by the supercomputer?

- Find another divergent series. How slowly does it diverge?

- Can you find a divergent series $\sum a_n$ such that $\lim_{n\to\infty} na_n = \infty$? What does this say about how slowly your series diverges?

$\mathbf{H}_n$ is almost never an integer

Suppose $n > 1$ and choose k such that

$$2^k \leq n < 2^{k+1}.$$

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k} + \cdots + \frac{1}{n}$$

Let M be the LCM of all the denominators except 2^k. That is,

$$M = \text{LCM}(1, 2, 3, \ldots, 2^k - 1, 2^k + 1, \ldots, n).$$

Note that M has a factor 2^{k-1} but not 2^k.

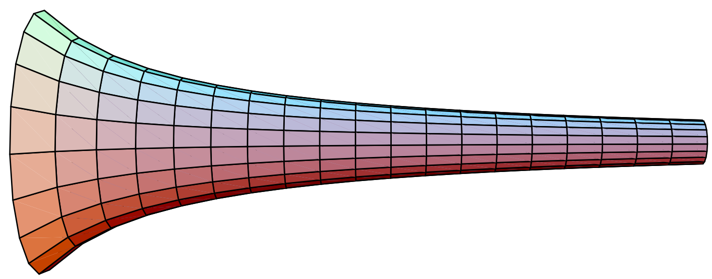
Multiply M and H_n.

$$\begin{aligned} M \cdot H_n &= M + \frac{M}{2} + \frac{M}{3} + \cdots + \frac{M}{2^k} + \cdots + \frac{M}{n} \\ &= \text{integer} + \frac{M}{2^k} + \text{integer}. \end{aligned}$$

Since $M/2^k$ is not an integer, $M \cdot H_n$ also cannot be an integer. Thus, H_n is not an integer.

Gabriel's wedding cake

Gabriel's horn is obtained by rotating the graph of $y = 1/x$, $1 \leq x < \infty$, about the x-axis.



- Gabriel's horn has finite volume but infinite surface area.

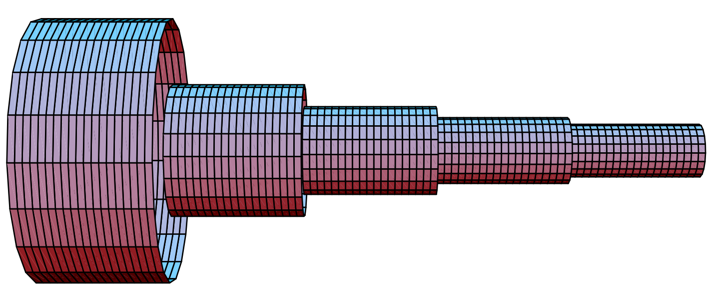
- It is sometimes said that the horn can be filled with paint, but cannot be painted.

Gabriel's wedding cake is a discrete analogue of Gabriel's horn.

Let f be the following piecewise-defined function:

$$f(x) = \begin{cases} 1, & 1 \leq x < 2 \\ 1/2, & 2 \leq x < 3 \\ & \ldots \\ 1/n, & n \leq x < n+1 \\ & \ldots \end{cases}$$

Now rotate the graph of $y = f(x)$, $1 \leq x < \infty$, about the x-axis.



- The cake has volume

$$V = \sum_{n=1}^{\infty} \pi \left(\frac{1}{n}\right)^2 (1) = \pi \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^3}{6}.$$

- The cake has lateral surface area

$$A = \sum_{n=1}^{\infty} 2\pi \left(\frac{1}{n}\right) (1) = 2\pi \sum_{n=1}^{\infty} \frac{1}{n}.$$

- Since the harmonic series diverges, Gabriel's wedding cake is a cake you can eat, but cannot frost.

Student explorations

- Research other paradoxical solids with finite volume but infinite surface area.

- Can you find a formula for the center of mass of any finite section of Gabriel's horn?

- What about for Gabriel's wedding cake?

The harmonic series diverges

Suppose the harmonic series converges with sum S.

$$\begin{aligned} S &= 1 + \left(\tfrac{1}{2} + \tfrac{1}{3}\right) + \left(\tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6}\right) \\ &\quad + \left(\tfrac{1}{7} + \tfrac{1}{8} + \tfrac{1}{9} + \tfrac{1}{10}\right) + \left(\tfrac{1}{11} + \cdots + \tfrac{1}{15}\right) \\ &\quad + \left(\tfrac{1}{16} + \cdots + \tfrac{1}{21}\right) + \cdots \\ &> 1 + \tfrac{2}{3} + \tfrac{3}{6} + \tfrac{4}{10} + \tfrac{5}{15} + \tfrac{6}{21} + \cdots \\ &= \tfrac{2}{2} + \tfrac{2}{3} + \tfrac{2}{4} + \tfrac{2}{5} + \tfrac{2}{6} + \tfrac{2}{7} + \cdots \\ &= 2 \textstyle\sum_{n=2}^{\infty} \tfrac{1}{n} \\ &= 2(S-1). \end{aligned}$$

The inequality $S > 2(S-1)$ implies $S < 2$.

Student explorations

- Why isn't this a convergence proof? Doesn't it show that the (increasing) sequence of partial sums is bounded above?

- Find another divergence proof that is based on grouping terms. How are the proofs similar? How are they different?

The leaning tower of lire

How far can a stack of equal-sized blocks be made to extend from the edge of a table?

- If each block has length 2 units, then n blocks can extend at most H_n units.
- This is a popular physics laboratory activity.
- Students are often surprised by this result.

Student explorations

- How many blocks would be required for the stack to extend 1000 units off the table? If each block had mass one gram, compare the mass of the stack to the estimated mass of the universe.

The harmonic series diverges

Proposition: For any natural number k,

$$\frac{1}{k} + \frac{1}{k+1} + \cdots + \frac{1}{3k} > 1.$$

Proof:

$$\begin{aligned} e^{\frac{1}{k}+\frac{1}{k+1}+\cdots+\frac{1}{3k}} &= e^{\frac{1}{k}} \cdot e^{\frac{1}{k+1}} \cdots e^{\frac{1}{3k}} \\ &> \left(1+\tfrac{1}{k}\right) \cdot \left(1+\tfrac{1}{k+1}\right) \cdots \left(1+\tfrac{1}{3k}\right) \\ &= \left(\tfrac{k+1}{k}\right) \cdot \left(\tfrac{k+2}{k+1}\right) \cdot \left(\tfrac{k+3}{k+2}\right) \cdots \left(\tfrac{3k+1}{3k}\right) \\ &= \tfrac{3k+1}{k} > 3. \end{aligned}$$

Corollary: The harmonic series diverges.

Student explorations

- Prove the proposition using the harmonic mean/arithmetic mean inequality.

- Notice that
$$\frac{1}{k} + \frac{1}{k+1} + \cdots + \frac{1}{3k} = H_{3k} - H_{k-1}.$$
Based on the proposition, we see that
$$\lim_{k\to\infty} (H_{3k} - H_{k-1}) \neq 0.$$
Explain how this allows us to conclude that the sequence $\{H_k\}$ does not converge.

- Use a CAS to evaluate $H_{3k} - H_k$ for several values of k. Evaluate this limit:
$$\lim_{k\to\infty} (H_{3k} - H_k)$$

Record Snowfall in Chicago

How often should Chicagoans expect record snowfall in January?

Assuming that the amount of snowfall in January of one year has no effect on the amount of snowfall in January of any subsequent year...

- The first year is a record year.

- The probability that the 2nd year is a record year is $\frac{1}{2}$. So, the expected number of record snowfalls in 2 years is $1 + \frac{1}{2}$.

- The probability that the 3rd year is a record year is $\frac{1}{3}$. So, the expected number of record snowfalls in 3 years is $1 + \frac{1}{2} + \frac{1}{3}$.

- In general, after n years of observation, we should expect H_n record years.

Inches of Snowfall for January, 1960-2004 Measured at O'Hare Airport-Chicago, IL (R denotes a record year)					
Year	Inches	Year	Inches	Year	Inches
1960	3.5 R	1975	3.5	1990	3.2
1961	3.0	1976	10.0	1991	11.1
1962	18.6 R	1977	7.2	1992	5.6
1963	16.8	1978	21.9	1993	15.2
1964	1.6	1979	34.3 R	1994	14.2
1965	11.7	1980	6.2	1995	13.1
1966	15.5	1981	2.0	1996	5.9
1967	25.1 R	1982	22.9	1997	no data
1968	10.4	1983	5.0	1998	no data
1969	3.7	1984	17.2	1999	29.6
1970	9.5	1985	18.9	2000	13.6
1971	10.0	1986	6.9	2001	1.5
1972	7.6	1987	17.3	2002	15.5
1973	0.5	1988	5.4	2003	4.3
1974	7.4	1989	0.4	2004	14.6
Chicago snowfall data obtained from the Illinois State Climatologist Office.					

Number of Illinois Tornadoes, 1956-2004 (R denotes a record year)					
Year	Tornadoes	Year	Tornadoes	Year	Tornadoes
1956	28 R	1973	63 R	1990	50
1957	42 R	1974	107 R	1991	32
1958	27	1975	46	1992	23
1959	37	1976	27	1993	34
1960	40	1977	33	1994	20
1961	34	1978	13	1995	76
1962	13	1979	12	1996	62
1963	11	1980	14	1997	29
1964	7	1981	33	1998	99
1965	28	1982	35	1999	64
1966	11	1983	14	2000	55
1967	40	1984	34	2001	21
1968	8	1985	15	2002	35
1969	10	1986	22	2003	120 R
1970	17	1987	22	2004	80
1971	16	1988	20		
1972	30	1989	15		
Illinois tornado data obtained from The Disaster Center.					

Student explorations

- Use your calculator to generate random numbers between 0 and 1. Keep track of record-breaking numbers. After n numbers, how many records did you find?

- About how many numbers would you expect to have to generate to obtain 10 records?

- Experiment with other sets of data.

The harmonic series diverges

For any divergent series, there is a much smaller divergent series.

Proposition: If $a_n > 0$, $\sum_{n=1}^{\infty} a_n$ diverges, and

$s_n = a_1 + a_2 + \cdots + a_n$, then $\sum_{n=1}^{\infty} \frac{a_{n+1}}{s_n}$ diverges.

Corollary 1: The harmonic series diverges.

Corollary 2: $\sum_{n=1}^{\infty} \frac{1}{nH_n}$ diverges (very, very slowly).

Student explorations

- How can the proposition be used to show that the harmonic series diverges?

- Just how slowly does $\sum_{n=1}^{\infty} \frac{1}{nH_n}$ diverge? How may terms are required for the partial sums to exceed 10?

- The proposition really shows that $\sum_{n=1}^{\infty} \frac{1}{(n+1)H_n}$ diverges. How is Corollary 2 obtained from this?

The Collector's Problem

If there is one toy per box, how many boxes of cereal should you expect to purchase if you want to collect a complete set of 6 toys?

Assuming that there is exactly one toy per box and that each toy is equally likely...

- The probability of getting one toy with the first box purchased is 1.

- Given that you have one toy, the probaility of getting a second (non-duplicate) toy with your next purchase is 5/6. So, the expected number of boxes you would need to purchase is 6/5.

- Given that you have two distinct toys, the probability of getting a third (non-duplicate) toy with your next purchase is 4/6. So, the expected number of boxes you would need to purchase is 6/4.

If we continue with this reasoning, you should expect to have a complete set after

$$1 + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + \frac{6}{1}$$

$$= 6\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}\right)$$

$$= 6\, H_6 \text{ purchases.}$$

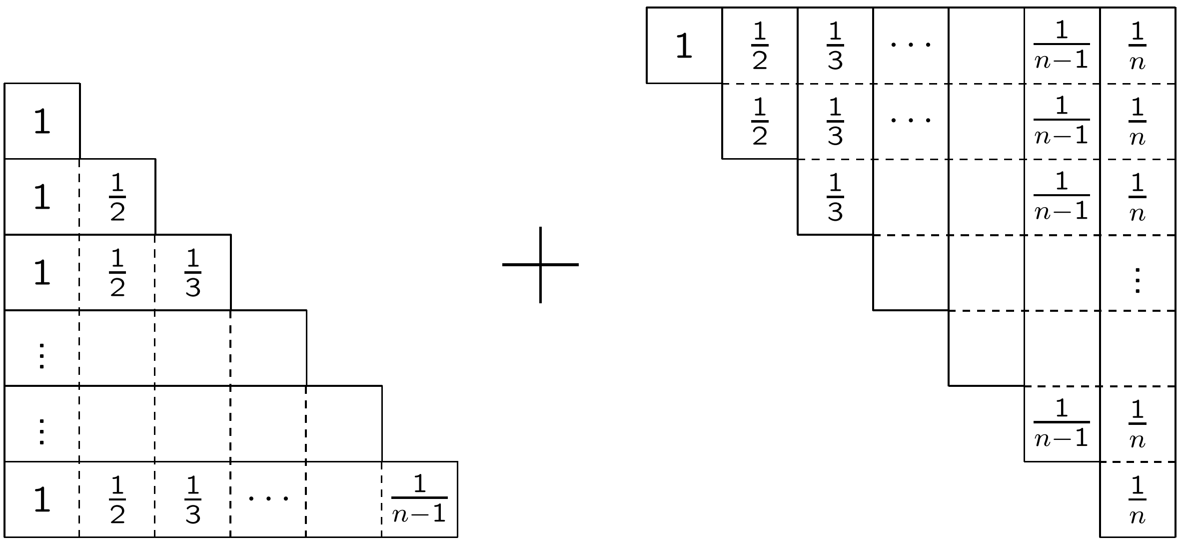
In general, the expected number of purchases necessary to complete one set of n objects is:

$$n\left(1 + \frac{1}{2} + \frac{1}{3} \cdots + \frac{1}{n}\right) = nH_n$$

Sums of partial sums

$$\sum_{k=1}^{n-1} H_k + n = nH_n$$

Proof



$$\sum_{k=1}^{n-1} H_k \qquad + \qquad n$$

Fit the shapes together for n rows of H_n

(Reference: Proofs Without Words II: More Exercises in Visual Thinking, Roger B. Nelsen)