Zeros of a Cubic Polynomial

The solution of the general cubic equation $ax^3 + bx^2 + cx + d = 0$ was first described in 1545 in Gerolamo Cardano's *Ars Magna*. Though Cardano discovered the solution, his method rested upon reducing the general equation to a simpler form. The reduction technique was discovered earlier by Niccolo Fontana (a.k.a. Tartaglia), and Tartaglia shared his technique with Cardano under an oath of secrecy. Cardano broke that oath. (Well, not really, but that's a long story.)

To solve the general cubic equation

$$ax^3 + bx^2 + cx + d = 0, \qquad a \neq 0$$

first let

$$p = \frac{3ac - b^2}{9a^2} \qquad q = \frac{9abc - 2b^3 - 27a^2 d}{54a^3} \qquad D = p^3 + q^2$$

Suppose p and q are real numbers.

- $D > 0 \Longrightarrow$ one real and two complex conjugate solutions
- $D = 0 \Longrightarrow$ three real solutions, at least two of which are equal
- $D < 0 \Longrightarrow$ three distinct real solutions

Now let

$$\alpha = q + \sqrt{D} \qquad \text{and} \qquad \beta = q - \sqrt{D}$$

The solutions are given by

$$\sqrt[3]{\alpha} + \sqrt[3]{\beta} - \frac{b}{3a}$$

$$-\frac{1}{2}\left(\sqrt[3]{\alpha} + \sqrt[3]{\beta}\right) - \frac{b}{3a} + \frac{\sqrt{3}}{2}\left(\sqrt[3]{\alpha} - \sqrt[3]{\beta}\right) i$$

$$-\frac{1}{2}\left(\sqrt[3]{\alpha} + \sqrt[3]{\beta}\right) - \frac{b}{3a} - \frac{\sqrt{3}}{2}\left(\sqrt[3]{\alpha} - \sqrt[3]{\beta}\right) i$$

If $D < 0$, it is easier to let $\theta = \cos^{-1}\left(\frac{q}{\sqrt{-p^3}}\right)$ and obtain the solutions from the following formulas:

$$2\sqrt{-p}\,\cos\left(\frac{\theta}{3}\right) - \frac{b}{3a}$$

$$2\sqrt{-p}\,\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right) - \frac{b}{3a}$$

$$2\sqrt{-p}\,\cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right) - \frac{b}{3a}$$