

Taylor's Theorem

If a function f is differentiable through order $n + 1$ in an interval I containing c , then, for each x in I , there exists a number z between x and c such that

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x - c)^{n+1}.$$

Taylor Series

If a function f has derivatives of all orders at $x = c$, then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x - c)^n = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + \cdots$$

is called the Taylor series for f at c . Moreover, if $c = 0$, then the Taylor series is called the Maclaurin series for f .

Convergence of Taylor Series

If a function f has derivatives of all orders in an open interval I centered at c , then the equality

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x - c)^n$$

holds if and only if there exists a z between x and c such that

$$\lim_{n \rightarrow \infty} R_n(x) = \lim_{n \rightarrow \infty} \frac{f^{(n+1)}(z)}{(n+1)!}(x - c)^{n+1} = 0$$

for every x in I .

Examples

1. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

2. $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

3. $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$