Applications of Integrals

1. **Arc Length**

   - **Function Form:** $v = f(u)$ represents a smooth curve on $a \le u \le b$.

   $$L = \int_a^b \sqrt{1 + [f'(u)]^2}\, du$$

   - **Parametric Form:** Smooth curve given by $x = f(t)$ and $y = g(t)$ for $a \le t \le b$.

   $$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

2. **Volume**

   - **Disc Method:** Solid made up of infinitesimal discs piled up along u-axis from $u = a$ to $u = b$. Disc at u has radius $f(u)$. Strips are perpendicular to rotation axis.

   $$V = \pi \int_a^b [f(u)]^2\, du$$

   - **Washer Method:** Solid made up of infinitesimal washers piled up along u-axis from $u = a$ to $u = b$. Washer at u has outside radius $R(u)$ and inside radius $r(u)$. Strips are perpendicular to rotation axis.

   $$V = \pi \int_a^b [(R(u))^2 - (r(u))^2]\, du$$

   - **Cylindrical Shells:** Solid made up of concentric infinitesimal cylindrical shells perpendicular to u-axis. Strips are parallel to rotation axis, piled up from $u = a$ to $u = b$.

   $$V = 2\pi \int_a^b \begin{pmatrix}\text{distance from rotation axis}\\ \text{to strip as a function of } u\end{pmatrix} \begin{pmatrix}\text{height of strip}\\ \text{as a function of } u\end{pmatrix} du$$

   - **Solids of Known Cross Section:** Solid made up of infinitesimal cross sections piled up along u-axis from $u = a$ to $u = b$. Cross section at u has area $A(u)$.

   $$V = \int_a^b A(u)\, du$$

3. **Fluid Force**

   - Fluid has constant weight-density w. Submerged plate made up of infinitesimal strips perpendicular to y-axis. Plate extends from $y = a$ to $y = b$.

   $$F = w \int_a^b \begin{pmatrix}\text{depth of strip}\\ \text{as a function of } y\end{pmatrix} \begin{pmatrix}\text{length of strip}\\ \text{as a function of } y\end{pmatrix} dy$$

4. **Work**

- **$\Delta W = (F)(\Delta x)$:** Usually used for rigid objects, springs, electric charges, leaky buckets, etc. Object moved from $x = a$ to $x = b$.

$$\Delta W = \begin{pmatrix}\text{force}\\ \text{at } x\end{pmatrix}\begin{pmatrix}\text{distance}\\ \text{increment}\end{pmatrix} = (F)(\Delta x) \quad \Longrightarrow \quad W = \int_a^b \begin{pmatrix}\text{force}\\ \text{at } x\end{pmatrix} dx$$

- **$\Delta W = (\Delta F)(x)$:** Usually used for nonrigid substances such as fluids, chains, gases, etc. Object moved from $x = a$ to $x = b$.

$$\Delta W = \begin{pmatrix}\text{force}\\ \text{increment}\end{pmatrix}\begin{pmatrix}\text{distance over which}\\ \text{force increment is applied}\end{pmatrix} = (\Delta F)(x)$$

$$\Longrightarrow \quad W = \int_a^b \begin{pmatrix}\text{distance over which } dF\\ \text{is applied as a function of } x\end{pmatrix}\begin{pmatrix}dF \text{ as a}\\ \text{function of } x\end{pmatrix}$$

5. **Moments and Mass**

- **Thin Rods:** Density at any u is $\delta(u)$. Rod extends from $u = a$ to $u = b$. Integrate with respect to u.
  (a) $dm = \delta(u)\, du$
  (b) Mass: $M = \int_a^b dm$
  (c) Moment about the origin: $M_0 = \int_a^b u\, dm$
  (d) Center of Mass: M_0/M
- **Thin Plates:** Density at any u is $\delta(u)$. Plate made up of infinitesimal strips perpendicular to u-axis extending from $u = a$ to $u = b$. Integrate with respect to u.
  (a) $dm = \begin{pmatrix}\text{distance across strip}\\ \text{as a function of } u\end{pmatrix}\delta(u)\, du$
  (b) Mass: $M = \int_a^b dm$
  (c) Moment about u-axis: $M_u = \int_a^b \begin{pmatrix}\text{signed distance from } u\text{-axis to}\\ \text{center of strip as a function of } u\end{pmatrix} dm$
  (d) Moment about v-axis: $M_v = \int_a^b \begin{pmatrix}\text{signed distance from } v\text{-axis to}\\ \text{center of strip as a function of } u\end{pmatrix} dm = \int_a^b u\, dm$
  (e) Center of mass: Coordinates given by $\bar{u} = M_v/M$ and $\bar{v} = M_u/M$

6. **Surface Area**

- Graph of $v = f(u)$, $a \le u \le b$, rotated about a horizontal or vertical axis.

$$S = 2\pi \int_a^b \begin{pmatrix}\text{distance between rotation axis and}\\ \text{graph of } f \text{ as a function of } u\end{pmatrix}\sqrt{1 + [f'(u)]^2}\, du$$