Ordinary Differential Equations

1. **Separable Equation**
   - Form: $\dfrac{dy}{dx} = f(x)\,g(y)$
   - Solution obtained from $\displaystyle\int \frac{dy}{g(y)} = \int f(x)\,dx + C$
2. **Exact Equation**
   - Form: $M(x,y)\,dx + N(x,y)\,dy = 0$ where $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$
   - Solution is defined implicitly by $F(x,y) = C$ where
   $$F(x,y) = \int M(x,y)\,dx + g(y)$$
   and
   $$F(x,y) = \int N(x,y)\,dy + h(x)$$
3. **1st Order, Linear Equation**
   - Form: $\dfrac{dy}{dx} + p(x)y = q(x)$
   - Let $\mu(x) = e^{\int p(x)\,dx}$
   - The solution follows from $\mu(x)y(x) = \displaystyle\int \mu(x)q(x)\,dx$.
4. **Bernoulli's Equation**
   - Form: $\dfrac{dy}{dx} + p(x)y = q(x)y^n$
   - Let $v = y^{1-n}$ to obtain the linear equation
   $$\frac{dv}{dx} + (1-n)p(x)v = (1-n)q(x)$$
5. **Exact after Integrating Factor**
   - Form: $M(x,y)\,dx + N(x,y)\,dy = 0$
     (a) If $(\partial M/\partial y - \partial N/\partial x) = 0$, the equation is exact.
     (b) If $(\partial M/\partial y - \partial N/\partial x) \div (-M) = g(y)$ is a function of only y, then $\mu(y) = e^{\int g(y)\,dy}$ is the integrating factor. Multiplication will make the equation exact.
     (c) If $(\partial M/\partial y - \partial N/\partial x) \div N = g(x)$ is a function of only x, then $\mu(x) = e^{\int g(x)\,dx}$ is the integrating factor. Multiplication will make the equation exact.
6. **Homogeneous Equation**
   - Form: $\dfrac{dy}{dx} = F\left(\dfrac{y}{x}\right)$
   - Substitute $v = y/x$ and $dy/dx = v + x\,dv/dx$.
   - The new equation is separable.

7. **Reducible to 1st-order (Type 1)**
   - Form: $F(x, y', y'') = 0$
   - Substitute $y' = u$ and $y'' = u'$.
   - The new equation involves only x, u, and u'. Solve for $u(x)$ and then for $y(x)$.

8. **Reducible to 1st-order (Type 2)**
   - Form: $F(y, y', y'') = 0$
   - Substitute $y' = u$ and $y'' = u\,\dfrac{du}{dy}$.
   - The new equation involves only y, u, and du/dy. Solve for $u(y)$ and then for $y(x)$.

9. **Euler's Method**
   - Given $\dfrac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$
   - $y(x_n) \approx y_n$ where

     $y_{n+1} = y_n + h\,f(x_n, y_n)$

     $x_{n+1} = x_n + h$ (h is the constant step size.)

10. **Improved Euler's Method**
    - Given $\dfrac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$
    - $y(x_n) \approx y_n$ where

      $y_{n+1} = y_n + \frac{h}{2}\left(f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})\right)$

      $y^*_{n+1} = y_n + h\,f(x_n, y_n)$

      $x_{n+1} = x_n + h$ (h is the constant step size.)

11. **Orthogonal Trajectories**
    - Given a one-parameter family of curves: $g(x, y) = c$
    - Find dy/dx. (Must not contain the constant c.)
    - Find a DE for the orthogonal trajectories by taking a negative reciprocal.
    - Solve the new DE.

12. **Homogenous, 2nd Order, Linear, Constant-Coefficient Equation**
    - Form: $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = 0$
    - The characteristic equation is $at^2 + bt + c = 0$.
    - Let $r = -b/2a$ and $\omega = \sqrt{|b^2 - 4ac|}/2a$.
      (a) If $b^2 - 4ac > 0$, the solution is $y(x) = c_1 e^{(r+\omega)x} + c_2 e^{(r-\omega)x}$.
      (b) If $b^2 - 4ac = 0$, the solution is $y(x) = c_1 e^{rx} + c_2 x e^{rx}$.
      (c) If $b^2 - 4ac < 0$, the solution is $y(x) = c_1 e^{rx} \cos \omega x + c_2 e^{rx} \sin \omega x$.

13. **2nd Order Cauchy-Euler Equation**
    - Form: $x^2\dfrac{d^2y}{dx^2} + bx\dfrac{dy}{dx} + cy = 0$
    - The substitution $x = e^t$ transforms the original equation to

$$\frac{d^2y}{dt^2} + (b-1)\frac{dy}{dt} + cy = 0.$$

    - Solve the new constant coefficient equation and resubstitute.

14. **Free Mechanical Vibrations**

- Model: $mx'' + bx' + kx = 0$

  (a) No damping if $b = 0$ (Simple harmonic motion)
  (b) Underdamped if $b^2 - 4mk < 0$ (Damped oscillations)
  (c) Overdamped if $b^2 - 4mk > 0$ (No oscillations)
  (d) Critically damped if $b^2 - 4mk = 0$ (No oscillations)

15. **Simple Harmonic Motion**

- $c_1 \cos \omega t + c_2 \sin \omega t = A \sin(\omega t + \phi), \quad A > 0$

  where

  $c_1 = A \sin \phi, \quad c_2 = A \cos \phi$

  $A = \sqrt{c_1^2 + c_2^2}$

  $\tan \phi = c_1 / c_2$

- Amplitude = A, Angular frequency = ω, Frequency = $f = \omega/(2\pi)$, Period = $T = 1/f$