Undetermined Coefficients for y'' + cy' + dy = g(x) (c and d are constants)

$$g(x)$$
 $y_p(x)$

(1)
$$p_n(x) = a_n x^n + \ldots + a_1 x + a_0$$
 $x^s P_n(x) = x^s (A_n x^n + \ldots + A_1 x + A_0)$

(2)
$$ae^{\alpha x}$$
 $x^s A e^{\alpha x}$

(3)
$$a \cos \beta x + b \sin \beta x$$
 $x^{s}(A \cos \beta x + B \sin \beta x)$

$$(4) p_n(x)e^{\alpha x} x^s P_n(x)e^{\alpha x}$$

(5)
$$p_n(x) \cos \beta x + q_m(x) \sin \beta x$$
, $x^s \{ P_N(x) \cos \beta x + Q_N(x) \sin \beta x \}$, where $q_m(x) = b_m x^m + \ldots + b_1 x + b_0$ where $Q_N(x) = B_n x^N + \ldots + B_1 x + B_0$ and $N = \max(n, m)$

(6)
$$ae^{\alpha x}\cos\beta x + be^{\alpha x}\sin\beta x$$
 $x^{s}(Ae^{\alpha x}\cos\beta x + Be^{\alpha x}\sin\beta x)$

(7)
$$p_n(x) e^{\alpha x} \cos \beta x + q_m(x) e^{\alpha x} \sin \beta x$$
 $x^s e^{\alpha x} \{ P_N(x) \cos \beta x + Q_N(x) \sin \beta x \},$ where $N = \max(n, m)$

The nonnegative integer s is chosen to be the least integer such that no term in $y_p(x)$ is a solution of the corresponding homogeneous equation y'' + cy' + dy = 0.

Variation of Parameters

If y_1 are y_2 are two linearly independent solutions of y'' + p(x)y' + q(x)y = 0, then a particular solution of y'' + p(x)y' + q(x)y = g(x) is $y = v_1y_1 + v_2y_2$, where

$$v_1(x) = \int \frac{-g(x)y_2(x)}{W[y_1, y_2](x)} dx,$$
 $v_2(x) = \int \frac{g(x)y_1(x)}{W[y_1, y_2](x)} dx,$

and $W[y_1, y_2](x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$.

2nd Solution from a 1st

If y_1 is a nonzero solution of y'' + p(x)y' + q(x)y = 0, then $y_2 = v \cdot y_1$, where

$$v(x) = \int \frac{1}{[y_1(x)]^2} \cdot e^{-\int p(x)dx} dx,$$

is also solution. Furthermore, y_1 and y_2 are linearly independent.