A first order equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n,$$

where P and Q are continous on an interval and n is a real number, is called a *Bernoulli equation*.

Divide both sides of the equation by y^n and then make the substitution $u = y^{1-n}$. This will transform the Bernoulli equation into the linear equation

$$\left(\frac{1}{1-n}\right)\frac{du}{dx} + P(x)u = Q(x).$$

Also note that $y = 0$ probably solves the original Bernoulli equation.

If the right-hand side of the equation

$$\frac{dy}{dx} = f(x, y)$$

can be expressed as a function of the ratio y/x alone, then we say the equation is *homogeneous*.

In this context, homogeneous means that the x and y variables have a balanced presence.

In a homogeneous equation, the substitution

$$u = \frac{y}{x}, \qquad \frac{dy}{dx} = u + x\frac{du}{dx}$$

will reduce it to a separable equation.

For certain 2nd order differential equations, the substitution $u = y'$ will reduce the the equation to a solvable 1st order equation.

For equations of the form

$$F(x, y', y'') = 0,$$

use the substitutions

$$y' = u \qquad y'' = u'$$

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The differential form $M(x,y)dx + N(x,y)dy$ is said to be *exact* on a rectangle R if it is the total differential of a function $F(x,y)$ on R. That is, $M(x,y)dx + N(x,y)dy$ is exact if

$$M(x,y) = \frac{\partial F}{\partial x} \qquad N(x,y) = \frac{\partial F}{\partial y}$$

for some function $F(x,y)$ on R.

If $M(x,y)dx + N(x,y)dy$ is an exact differential form, then the differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

is called an *exact equation*.

Test for exactness

Suppose the first partial derivatives of $M(x, y)$ and $N(x, y)$ are continuous in a rectangle R. Then

$$M(x, y)dx + N(x, y)dy = 0$$

is an exact differential equation if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

for all (x, y) in R.