

Bernoulli equations

A first-order equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n,$$

where P and Q are continuous on an interval and n is a real number, is called a *Bernoulli equation*.

Divide both sides of the equation by y^n and then make the substitution $u = y^{1-n}$. This will transform the Bernoulli equation into the linear equation

$$\left(\frac{1}{1-n}\right) \frac{du}{dx} + P(x)u = Q(x).$$

Also note that $y(x) = 0$ may also solve the original Bernoulli equation.

Homogeneous equations

If the right-hand side of the equation

$$\frac{dy}{dx} = f(x, y)$$

can be expressed as a function of the ratio y/x alone, then we say the equation is *homogeneous*.

In this context, homogeneous means that the terms of the equation are dimensionally balanced.

In a homogeneous equation, the substitution

$$u = \frac{y}{x}, \quad \frac{dy}{dx} = u + x \frac{du}{dx}$$

will reduce it to a separable equation.

Reducible to 1st-order

For certain 2nd-order differential equations, the substitution $u = y'$ will reduce the equation to a solvable 1st-order equation.

For equations of the form

$$F(x, y', y'') = 0,$$

use the substitutions

$$y' = u \quad y'' = u'$$

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Exact equations

The differential form $M(x, y)dx + N(x, y)dy$ is said to be *exact* on a rectangle R if it is the total differential of a function $F(x, y)$ on R . That is, $M(x, y)dx + N(x, y)dy$ is exact if

$$M(x, y) = \frac{\partial F}{\partial x} \quad N(x, y) = \frac{\partial F}{\partial y}$$

for some function $F(x, y)$ on R .

If $M(x, y)dx + N(x, y)dy$ is an exact differential form, then the differential equation

$$M(x, y)dx + N(x, y)dy = 0$$

is called an *exact equation*.

Test for exactness

Suppose the first partial derivatives of $M(x, y)$ and $N(x, y)$ are continuous in a rectangle R . Then

$$M(x, y)dx + N(x, y)dy = 0$$

is an exact differential equation if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

for all (x, y) in R .