Series solutions around singular points

Theorem: Any differential equation of the form

$$y'' + \frac{p(x)}{x} y' + \frac{q(x)}{x^2} y = 0,$$

where $p(x)$ and $q(x)$ are analytic at $x = 0$, has at least one solution of the form

$$y(x) = x^s \sum_{n=0}^{\infty} a_n x^n,$$

where s is a real or complex constant chosen so that $a_0 \neq 0$.

Solution procedure:

- Write

$$p(x) = \sum_{n=0}^{\infty} p_n x^n \qquad q(x) = \sum_{n=0}^{\infty} q_n x^n.$$

- Rewrite the original equation as

$$x^2 y'' + xp(x)y' + q(x)y = 0,$$

  and use our usual series solution method.

- The term involving a_0 should give rise to the *indicial equation*

$$s^2 + (p_o - 1)s + q_0 = 0.$$

- Solve for s and find the series solution(s) associated with each particular s.

- If the indicial equation has distinct solutions that do not differ by an integer, then two linearly independent solutions have the given form. Otherwise, you might not get two solutions from this approach.