Laplace Transform

Let $f(t)$ be a function defined on $[0, \infty)$. The *Laplace transform* of f is the function F such that

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt,$$

for all s for which the improper integral converges.

Other notation for the Laplace transform is $F = \mathscr{L}\{f\}$.

Theorem: Let f_1 and f_2 be functions whose Laplace transforms exist for $s > a$, and let c_1 and c_2 be constants. Then

$$\mathscr{L}\{c_1 f_1 + c_2 f_2\} = c_1 \mathscr{L}\{f_1\} + c_2 \mathscr{L}\{f_2\},$$

for $s > a$.

Definition: A function f is said to be of *exponential order* α if there exist positive constants T and M such that

$$|f(t)| \le Me^{\alpha t},$$

for all $t \ge T$.

Theorem: If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order α, then $F = \mathscr{L}\{f\}$ exists for $s > \alpha$.

Furthermore, $F(s) \to 0$ and $s \to \infty$.

Theorem: Suppose f and g are piecewise continuous for $t \geq 0$ and are of exponential order. Let F and G be their corresponding Laplace transforms. If $F(s) = G(s)$ for all $s > c$, then $f(t) = g(t)$ wherever on $[0, \infty)$ that both are continuous.

Inverse Laplace Transform

Given a function $F(s)$, if there is a function $f(t)$ that is continuous on $[0, \infty)$ and satisfies $\mathscr{L}\{f\} = F$, then we say that f is the inverse Laplace transform of F.

Other notation for the inverse Laplace transform is $f = \mathscr{L}^{-1}\{F\}$.