Convolution

Definition: Suppose f and g are piecewise continuous for $t \geq 0$. The *convolution* of f and g, denoted by $f * g$, is given by

$$(f * g)(t) = \int_0^t f(\tau) g(t - \tau)\, d\tau.$$

Theorem: Suppose f and g are piecewise continuous for $t \geq 0$ and of exponential order α. Then, for $s > \alpha$, $\mathscr{L}\{f * g\}$ exists. Furthermore,

$$\mathscr{L}\{f(t) * g(t)\} = \mathscr{L}\{f(t)\} \cdot \mathscr{L}\{g(t)\} = F(s) \cdot G(s)$$

and

$$\mathscr{L}^{-1}\{F(s) \cdot G(s)\} = f(t) * g(t).$$

Derivatives of transforms

Theorem: Suppose f is piecewise continuous for $t \geq 0$ and of exponential order α. Then, for $s > \alpha$,

$$\mathscr{L}\{t^n f(t)\}(s) = (-1)^n F^{(n)}(s).$$

It also follows that

$$\mathscr{L}^{-1}\{F(s)\}(t) = -\frac{1}{t}\mathscr{L}^{-1}\{F'(s)\}(t).$$

Transforms of integrals

Theorem: Suppose f is piecewise continuous for $t \geq 0$ and of exponential order α. Then, for $s > \alpha$,

$$\mathscr{L}\left\{\int_0^t f(\tau)d\tau\right\}(s) = \frac{1}{s}\mathscr{L}\{f(t)\}(s) = \frac{F(s)}{s}$$

and equivalently,

$$\mathscr{L}^{-1}\left\{\frac{F(s)}{s}\right\}(t) = \int_0^t f(\tau)\,d\tau.$$

Integrals of transforms

Theorem: Suppose f is piecewise continuous for $t \geq 0$ and of exponential order α. Further suppose that $\lim_{t \to 0^+} \frac{f(t)}{t}$ exists and is finite. Then, for $s > \alpha$,

$$\mathscr{L}\left\{\frac{f(t)}{t}\right\}(s) = \int_s^\infty F(\sigma)\, d\sigma$$

and equivalently,

$$f(t) = \mathscr{L}^{-1}\{F(s)\}(t) = t\,\mathscr{L}^{-1}\left\{\int_s^\infty F(\sigma) d\sigma\right\}(t).$$