Ex FIND A POWER SERIES SOLUTION CENTERED AT $X = 0$:

$$y' + 2xy = 0.$$

BEFORE STARTING, NOTICE THE COEFFICIENT IS ANALYTIC AT $X = 0$. THAT IS, $X = 0$ IS AN ORDINARY POINT.

SOLUTION...

ASSUME A SOLUTION OF THE FORM $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

POWER SERIES CENTERED AT $X = 0$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \Rightarrow y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

SUBSTITUTE INTO EQUATION.

$$0 = y' + 2xy = \sum_{n=1}^{\infty} n a_n x^{n-1} + 2x \sum_{n=0}^{\infty} a_n x^n$$

DISTRIBUTE

$$= \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 2x a_n x^n$$

$$= \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 2 a_n x^{n+1}$$

NOW WE'D LIKE TO SHIFT THE STARTING INDEX SO THAT EACH EXPRESSION HAS THE SAME EXPONENT ON X.

TO SHIFT, WE REINDEX.

IN THE FIRST SUM, REPLACE EVERY n BY $n+1$

TO GET $\sum_{n=0}^{\infty} (n+1)a_{n+1} x^n$

IN THE SECOND SUM, REPLACE EVERY n BY $n-1$

TO GET $\sum_{n=1}^{\infty} 2a_{n-1} x^n$.

SO OUR EQUATION IS NOW

$$\sum_{n=0}^{\infty} (n+1)a_{n+1} x^n + \sum_{n=1}^{\infty} 2a_{n-1} x^n = 0$$

NOW MAKE THE STARTING INDEX THE SAME

BY WRITING THE FIRST EXPRESSION LIKE

THIS: $\underbrace{\underbrace{(n=0 \text{ TERM})}_{(0+1)a_{0+1}x^0}}_{a_1} + \sum_{n=1}^{\infty} (n+1)a_{n+1} x^n$

NOW OUR EQUATION IS

$$a_1 + \underbrace{\sum_{n=1}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=1}^{\infty} 2a_{n-1} x^n}_{\text{WRITE AS A SINGLE SUM.}} = 0$$

THIS WAS THE POINT OF THE REINDEXING.

$$a_1 + \sum_{n=1}^{\infty} \left[(n+1) a_{n+1} + 2a_{n-1} \right] x^n = 0$$

TO BE ZERO, ALL COEFFICIENTS MUST BY ZERO.

$$\Downarrow$$

$$a_1 = 0 \, , \quad (n+1) a_{n+1} + 2a_{n-1} = 0$$

$$a_1 = 0 \, , \quad a_{n+1} = \frac{-2}{n+1} a_{n-1}$$ ← THIS IS CALLED A RECURRENCE RELATION.

LET a_0 BE OUR ARBITRARY CONSTANT OF INTEGRATION, AND LET'S WRITE OUT SOME OF THESE a_n'S.

$a_0 = \text{CONST.}$

$a_1 = 0$

$a_2 = \frac{-2}{2} a_0 = -a_0$

$a_3 = -\frac{2}{3} a_1 = 0$

$a_4 = \frac{-2}{4} a_2 = \frac{1}{2} a_0$

$a_5 = \frac{-2}{5} a_3 = 0$

$a_6 = \frac{-2}{6} a_4 = -\frac{1}{3} \cdot \frac{1}{2} a_0$

$a_7 = \frac{-2}{7} a_5 = 0$

$a_8 = \frac{-2}{8} a_6 = \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{2} a_0$

IN GENERAL, FOR $n = 1, 2, 3, \ldots$

$a_{2n+1} = 0$, $a_{2n} = \frac{(-1)^n}{n!} a_0$

SO OUR POWER SERIES HAS THE FORM

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + 0x - a_0 x^2 + 0x^3$$

$$+ \frac{1}{2} a_0 x^4 + 0x^5 - \frac{1}{3\cdot 2} a_0 x^6$$

$$+ 0x^7 + \frac{1}{4\cdot 3\cdot 2} a_0 x^8 + \cdots$$

$$= a_0 \left(1 - x^2 + \frac{1}{2} x^4 - \frac{1}{3\cdot 2} x^6 + \frac{1}{4\cdot 3\cdot 2} x^8 - \cdots \right)$$

$$= a_0 \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}$$

IT IS EASY TO CHECK USING THE RATIO TEST (FROM CALC II) THAT THE POWER SERIES CONVERGES ON $(-\infty, \infty)$.

AND, IF WE RECOGNIZE THAT $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$, (SEE THE COMMON SERIES SHEET)

THEN WE SEE THAT

$$\boxed{y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = a_0 e^{-x^2}.}$$

HERE IS ONE TO TRY ON YOUR OWN.

USE A POWER SERIES TO SOLVE

$$(x-3)y' + 2y = 0.$$