Ex Use a power series to solve $2y'' + xy' + y = 0$.

Solution

Since $x = 0$ is an ordinary point, we can center the power series at $x = 0$. Furthermore, since the equation has no singular points, we expect our power series solution to have an infinite radius of convergence. (See lecture slides.)

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute ...

$$2y'' + xy' + y = \underbrace{\sum_{n=2}^{\infty} 2n(n-1) a_n x^{n-2}}_{\text{Replace } n \text{ with } n+2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} 2(n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n$$

Pull off the $n = 0$ terms so all start with $n = 1$ and combine.

$$= 2(2)(1)a_2 + a_0 + \sum_{n=1}^{\infty}\left[2(n+2)(n+1)a_{n+2} + na_n + a_n\right]x^n$$

$$= 4a_2 + a_0 + \sum_{n=1}^{\infty}\left[2(n+2)(n+1)a_{n+2} + (n+1)a_n\right]x^n$$

$$= 0$$

THEREFORE,

$$4a_2 + a_0 = 0$$

$$2(n+2)(n+1)a_{n+2} + (n+1)a_n = 0$$

- OR -

$$a_2 = -\frac{1}{4}a_0 \quad \text{AND} \quad a_{n+2} = \frac{-1}{2(n+2)}a_n \quad \text{FOR } n = 1, 2, 3, \ldots$$

LET a_0 AND a_1 BE OUR TWO ARBITRARY CONSTANTS OF INTEGRATION.

$$a_2 = -\frac{1}{4}a_0$$

$$a_3 = -\frac{1}{2\cdot 3}a_1$$

$$a_4 = -\frac{1}{2(4)}a_2 = -\frac{1}{2(4)}\cdot\left(-\frac{1}{4}\right)a_0$$

$$a_5 = -\frac{1}{2(5)}a_3 = -\frac{1}{2(5)}\cdot\left(-\frac{1}{2\cdot 3}\right)a_0$$

IT MAY TAKE A FEW MORE TERMS TO OBSERVE THE FOLLOWING PATTERN:

WITH a_0 & a_1 ARBITRARY CONSTANTS,

$$a_{2n} = \frac{(-1)^n a_0}{2^n (2\cdot 4\cdot 6 \cdots 2n)}, \quad n = 1, 2, 3, 4, \ldots$$

$$a_{2n+1} = \frac{(-1)^n a_1}{2^n (1\cdot 3\cdot 5 \cdots (2n+1))}, \quad n = 1, 2, 3, 4, \ldots$$

OUR SOLUTION IS

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{WITH THOSE } a_n\text{'s} \ldots$$

$$y(x) = \underbrace{a_0 + a_0 \sum_{n=1}^{\infty} a_{2n} x^{2n}}_{\text{EVEN-POWERED TERMS}} + \underbrace{a_1 x + a_1 \sum_{n=1}^{\infty} a_{2n+1} x^{2n+1}}_{\text{ODD-POWERED TERMS}}$$

$$= \underbrace{a_0 + a_0 \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^n (2\cdot 4\cdot 6 \cdots 2n)}}_{\text{CALL THIS } y_1(x)} + \underbrace{a_1 x + a_1 \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2^n (1\cdot 3\cdot 5 \cdots (2n+1))}}_{\text{CALL THIS } y_2(x).}$$

$y_1(x)$ AND $y_2(x)$ ARE LINEARLY INDEPENDENT SOLUTIONS OF THE EQUATION.

IT IS NOT DIFFICULT TO CHECK BY USING THE RATIO TEST (CALC II) THAT BOTH $y_1(x)$ AND $y_2(x)$ CONVERGE ABSOLUTELY FOR ALL X.

(RECALL THAT WE ALREADY KNEW THE SOLUTION WOULD HAVE AN INFINITE RADIUS OF CONVERGENCE.)