Ex Use a power series to solve $y'' - xy' - x^2y = 0$.

Assume $y(x) = \sum_{n=0}^{\infty} a_n x^n$. The radius of convergence will be ∞.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute ...

$$y'' - xy' - x^2 y = \underbrace{\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}}_{\text{Replace } n \text{ by } n+2} - \sum_{n=1}^{\infty} n a_n x^n - \underbrace{\sum_{n=0}^{\infty} a_n x^{n+2}}_{\text{Replace } n \text{ by } n-2}$$

$$= \underbrace{\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n}_{\text{Pull off terms } n=0 \text{ \& } n=1} - \sum_{n=2}^{\infty} a_{n-2} x^n$$

And combine.

$$= 2a_2 + 6a_3 x - a_1 x + \sum_{n=2}^{\infty} \left[(n+2)(n+1) a_{n+2} - n a_n - a_{n-2} \right] x^n$$

$$= 0$$

THEREFORE,

$$a_2 = 0$$

$$6a_3 - a_1 = 0$$

$$(n+2)(n+1) a_{n+2} - n a_n - a_{n-2} = 0$$

- OR -

$$a_2 = 0, \quad a_3 = \frac{1}{6} a_1, \quad a_{n+2} = \frac{n a_n + a_{n-2}}{(n+2)(n+1)}.$$

THIS IS A THREE-TERM RECURRENCE RELATION.

LET a_0 AND a_1 BE OUR ARBITRARY CONSTANTS OF INTEGRATION.

a_0, a_1 ARBITRARY

$$a_2 = 0$$

$$a_3 = \frac{1}{6} a_1$$

$$a_4 = \frac{2a_2 + a_0}{12}$$

$$a_5 = \frac{3a_3 + a_1}{20}$$

$$a_6 = \frac{4a_4 + a_2}{30}$$

$$a_7 = \frac{5a_5 + a_3}{42}$$

$$a_8 = \frac{6a_6 + a_4}{56}$$

$$\vdots$$

TO OBTAIN OUR TWO LINEARLY INDEPENDENT SOLUTIONS, LET'S DO THIS...

1^{ST} SOLUTION: LET $a_0 = 1$, $a_1 = 0$

SO THAT
$$a_2 = 0$$
$$a_3 = 0$$
$$a_4 = \frac{1}{12}$$
$$a_5 = 0$$
$$a_6 = \frac{1}{90}$$
$$a_7 = 0$$
$$a_8 = \frac{\frac{6}{90} + \frac{1}{12}}{56} = \frac{3}{1120}$$
$$\vdots$$

$$\boxed{y_1(x) = 1 + \frac{1}{12}x^4 + \frac{1}{90}x^6 + \frac{3}{1120}x^8 + \cdots}$$

2^{ND} SOLUTION: LET $a_0 = 0$, $a_1 = 1$

SO THAT

$$a_2 = 0$$

$$a_3 = \frac{1}{6}$$

$$a_4 = 0$$

$$a_5 = \frac{3}{40}$$

$$a_6 = 0$$

$$a_7 = \frac{\frac{3}{8} + \frac{1}{6}}{42} = \frac{13}{1008}$$

$$\vdots$$

$$\boxed{y_2(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{13}{1008}x^7 + \cdots}$$

NOW WE HAVE

$$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

WHERE y_1 & y_2 ARE LINEARLY INDEPENDENT.