Some Properties of Power Series

At the end of the last section, we started thinking about using power series to represent functions. In this section, we continue that theme by looking at some results that describe how power series, and the functions they represent, can be combined and manipulated.

Theorem 1

Suppose that the two power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ converge to the functions f and g, respectively, on the common interval I.

1. The power series $\sum_{n=0}^{\infty}(a_n x^n \pm b_n x^n)$ converges to $f \pm g$ on I.
2. For any non-negative integer m and any real number c, the power series $\sum_{n=0}^{\infty} cx^m a_n x^n$ converges to $cx^m f(x)$ on I.
3. For any non-negative integer m and any real number c, the power series $\sum_{n=0}^{\infty} a_n (cx^m)^n$ converges to $f(cx^m)$, as long as cx^m is in I.

Example 1

Find a power series $f(x) = \dfrac{3x-1}{x^2-1}$.

After computing the partial fraction decomposition, you will find that $f(x)$ can be written

$$f(x) = \frac{2}{x+1} + \frac{1}{x-1}.$$

Let's rewrite $f(x)$ one more time...

$$f(x) = \frac{2}{1-(-x)} - \frac{1}{1-x}.$$

Now it should be obvious that $f(x)$ is the difference of two geometric series, both of which converge on the interval $(-1, 1)$:

$$\frac{2}{1-(-x)} - \frac{1}{1-x} = \sum_{n=0}^{\infty} 2(-x)^n - \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty}(2(-1)^n - 1)x^n.$$

Therefore,

$$f(x) = \sum_{n=0}^{\infty}(2(-1)^n - 1)x^n = 1 - 3x + x^2 - 3x^3 + x^4 - 3x^5 + \cdots, \qquad -1 < x < 1. \qquad \diamondsuit$$

Example 2

Find a function that is represented by the power series $\sum_{n=0}^{\infty} 2^{n+1} x^n$.

In order to better recognize this series, let's rewrite...

$$\sum_{n=0}^{\infty} 2^{n+1} x^n = \sum_{n=0}^{\infty} 2\,(2x)^n.$$

Now we recognize a geometric series with $a = 2$ and $r = 2x$. The series converges if and only if $|2x| < 1$ or $|x| < 1/2$.

So we have it:

$$f(x) = \sum_{n=0}^{\infty} 2^{n+1} x^n = \frac{2}{1-2x}, \qquad -\frac{1}{2} < x < \frac{1}{2}. \qquad \diamondsuit$$

Theorem 2

Suppose the power series $\sum_{n=0}^{\infty} a_n\,(x-c)^n$ converges on the interval $(c-R, c+R)$ for some positive radius of convergence R, and let f be the function defined by the power series.

1. f is differentiable on the interval $(c-R, c+R)$, and f' can be obtained by differentiating the power series term-by-term:

$$f'(x) = \sum_{n=1}^{\infty} n a_n\,(x-c)^{n-1}, \qquad c-R < x < c+R.$$

2. f is integrable on the interval $(c-R, c+R)$, and $\int f(x)dx$ can be obtained by integrating the power series term-by-term:

$$\int f(x)dx = K + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1}, \qquad c-R < x < c+R,$$

   where K is the arbitrary constant of integration.

Term-by-term differentiation and integration do not guarantee anything about the behavior of the series at the endpoints of the interval of convergence. (You must check separately.)

Example 3

Use term-by-term differentiation to obtain a power series representation for $g(x) = \dfrac{1}{(1-x)^2}$ from that of $f(x) = \dfrac{1}{1-x}$.

First, $f(x)$ is the sum of the geometric series with $a = 1$ and $r = x$:

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots, \qquad -1 < x < 1.$$

Next, $g(x) = f'(x)$, so a series for g can be obtained by differentiating the series for f:

$$g(x) = f'(x) = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=1}^{\infty} n x^{n-1}, \qquad -1 < x < 1. \qquad \diamondsuit$$

Example 4

Use the result from example 4 to find the exact value of the sum of the series $\sum_{n=0}^{\infty} \dfrac{n+1}{4^n}$.

Rewrite the result above so that the index starts at $n = 0$:

$$g(x) = \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n.$$

Now simply let $x = 1/4$, which is inside the interval of convergence of g:

$$g(1/4) = \frac{1}{(1-\frac{1}{4})^2} = \sum_{n=0}^{\infty}(n+1)\left(\frac{1}{4}\right)^n = \sum_{n=0}^{\infty}\frac{n+1}{4^n}.$$

Since, $g(1/4) = 16/9$, the sum of the series is $16/9$. ♢

Example 5

Find a power series for $f(x) = \ln(x+1)$.

Since $f'(x) = \dfrac{1}{1+x}$, we can obtain a series for $f(x)$ by integrating the series the $f'(x)$.

$$f'(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n, \qquad -1 < x < 1.$$

Therefore,

$$f(x) = \ln(x+1) = C + \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1} = C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$$

Since $f(0) = 0$, the constant of integration must satisfy $C = 0$.

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots, \qquad -1 < x < 1.$$

It turns out that if we check for convergence at the interval endpoints, we will find that the interval of convergence of the series for $\ln(x+1)$ is actually $(-1, 1]$. ♢