Power Series

Recall that we defined a *geometric series* as one of the form $\sum_{n=0}^{\infty} ar^n$. At that time, we were treating a and r as constants. In this section, we will start thinking about what happens if a and r are variable.

Definition 1

A power series centered at $x = c$ is a series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n,$$

where x is a variable and $\{a_n\}_{n=0}^{\infty}$ is a sequence of constants.

For the purposes of power series, we stipulate that $x^0 = 1$ and $(x-c)^0 = 1$ even when $x = 0$ and $x = c$, respectively. (Normally we consider 0^0 to be indeterminate.)

Example 1

The series $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ is a power series centered $x = 0$.

For any x, this series is also a geometric series. Therefore, we know that if $|x| < 1$, the series converges and

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}. \qquad \diamondsuit$$

Example 2

The series $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ is another power series centered at $x = 0$. $\diamondsuit$

Example 3

The series $\sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^n}{n+1}$ is power series centered at $x = 1$. $\diamondsuit$

Theorem

For a power series centered at $x = c$, exactly one of the following is true:

1. The series converges only at $x = c$. It diverges for $x \neq c$.
2. The series converges absolutely for all x.
3. There exists a real number $R > 0$ such that the series converges absolutely if $|x - c| < R$ and diverges if $|x - c| > R$. At the values of x for which $|x - c| = R$, the series may converge or diverge.

Definition 2

The number R in case 3 of the theorem above is called the radius of convergence of the power series. In case 1, we say $R = 0$; and in case 2, we say $R = \infty$. The set of all x-values for which a power series converges is called its interval of convergence.

A typical approach to finding the radius and interval of convergence of a power series is to apply a convergence test that includes some kind of inequality condition. The typical choices are the ratio, root, and geometric series tests. The next examples will illustrate the ideas.

Example 4

Find the radius and interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

By the ratio test, this series converges absolutely when

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} < 1.$$

For any value of x, $\frac{|x|}{n+1} \to 0$ as $n \to \infty$, so the limit is zero. Zero is always less than one! This series converges absolutely for every number x. The radius of converges is ∞, and the interval of convergence is $(-\infty, \infty)$. $\diamondsuit$

Example 5

Find the radius and interval of convergence of the power series $\sum_{n=0}^{\infty} n!\, x^n$.

Let's use the ratio test to test for convergence:

$$\lim_{n\to\infty} \left| \frac{(n+1)!\, x^{n+1}}{n!\, x^n} \right| = \lim_{n\to\infty} (n+1)\, |x|.$$

For any $x \neq 0$, the limit is ∞. This series diverges everywhere except for $x = 0$. The radius of convergence is 0. The interval of convergence is really not an interval at all, it is the single number $x = 0$. $\diamondsuit$

Example 6

Find the radius and interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{(x-1)^n}{(n+1)\, 2^n}$.

Again, we will use the ratio test.

$$\lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{(n+2)\, 2^{n+1}} \cdot \frac{(n+1)\, 2^n}{(x-1)^n} \right| = \lim_{n\to\infty} \frac{|x-1|}{2} \frac{n+1}{n+2} = \frac{|x-1|}{2}$$

The series converges if $\frac{|x-1|}{2} < 1$ or $|x-1| < 2$. So the radius of convergence is 2.

To find the interval of convergence, we solve the inequality $|x-1| < 2$.

$$|x-1| < 2 \iff -2 < x-1 < 2 \iff -1 < x < 3.$$

We have so far established that the interval of convergence is $(-1, 3)$. Notice that this is the open interval centered at $x = 1$ with radius 2. But according to part 3 of the theorem, we have more to consider. The series may actually converge at the interval endpoints. We must check!

For $x = -1$, the series is

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{(n+1)2^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)},$$

which converges by the alternating series test.

For $x = 3$, the series is

$$\sum_{n=0}^{\infty} \frac{2^n}{(n+1)2^n} = \sum_{n=0}^{\infty} \frac{1}{(n+1)},$$

which is the divergent harmonic series.

Therefore, when all is said and done, the interval of convergence is $[-1, 3)$. $\diamondsuit$

Comments

1. For the interval of convergence, you will always have to individually check the interval endpoints. This is an important step, but it is easy to overlook.
2. A power series is a function whose domain is its interval of convergence.
3. When a power series describes a function, it may be that the function can be written in a more familiar way. For instance, see example 1 above. The final examples further illustrate this idea.

Example 7

Use a power series to represent $f(x) = \dfrac{1}{1+x^3}$.

The form of $f(x)$ leads us to think about geometric series. Notice that

$$\frac{1}{1+x^3} = \frac{1}{1-(-x^3)}.$$

It follows that $f(x)$ is the sum of a geometric series with $a = 1$ and $r = -x^3$:

$$\frac{1}{1+x^3} = \sum_{n=0}^{\infty}(-x^3)^n = 1 - x^3 + x^6 - x^9 + \cdots.$$

This geometric series converges when $|-x^3| < 1$, which is precisely when $|x| < 1$. So the interval of convergence is $(-1, 1)$. (It is easy to see that the series diverges at the interval endpoints.)

$\diamondsuit$

Example 8

Use a power series to represent $f(x) = \dfrac{x^3}{2-x}$.

Thinking of geometric series again, let's rewrite $f(x)$:

$$f(x) = \frac{x^3}{2-x} = \left(\frac{x^3}{2}\right)\left(\frac{1}{1-\frac{x}{2}}\right).$$

Now $f(x)$ is the product of $x^3/2$ and a geometric series with $a = 1$ and $r = x/2$:

$$\left(\frac{x^3}{2}\right)\left(\frac{1}{1-\frac{x}{2}}\right) = \frac{x^3}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{x^{n+3}}{2^{n+1}}.$$

This series converges when $|x/2| < 1$ or $|x| < 2$. The radius of convergence is 2, and the interval of convergence is $(-2, 2)$. Since we already know that geometric series diverge when $r = 1$, there is no need (in this case) to check interval endpoints. ♢