Taylor Series

In Calculus I, we learned how to use a linear function to approximate a more general function. The idea goes by lots of different names: linearization, tangent line approximation, standard linear approximation, etc. For a quick review, see the lecture 20 notes posted at http://stevekifowit.com/archives/M131/.

The linearization of f at $x = c$ is the function

$$P_1(x) = f(c) + f'(c)(x - c).$$

The approximation $f(x) \approx P_1(x)$ is the standard linear approximation. It is easy to verify that P_1 is the unique 1st degree polynomial satisfying

$$P_1(c) = f(c), \qquad P_1'(c) = f'(c).$$

In other words, P_1 is the only linear function that matches f and f' at $x = c$.

If you're thinking like a mathematician, you have a natural question: What if we also want f'' to match at $x = c$? Well, then we'd need the quadratic function

$$P_2(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2}(x - c)^2.$$

P_2 is the unique polynomial of degree ≤ 2 satisfying

$$P_2(c) = f(c), \qquad P_2'(c) = f'(c), \qquad P_2''(c) = f''(c).$$

The approximation $f(x) \approx P_2(x)$ is the standard quadratic approximation.

We can continue indefinitely (as long as f has enough derivatives). The polynomial

$$P_n(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \frac{f'''(c)}{3!}(x - c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n$$

is the unique polynomial of degree $\leq n$ satisfying

$$P_n(c) = f(c), \qquad P_n'(c) = f'(c), \qquad P_n''(c) = f''(c), \qquad \ldots, \qquad P_n^{(n)}(c) = f^{(n)}(c).$$

Definition 1

Suppose f has n derivatives at $x = c$, then the nth Taylor polynomial for f at $x = c$ is the polynomial (of degree $\leq n$) given by

$$P_n(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \frac{f'''(c)}{3!}(x - c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n.$$

If $c = 0$, the Taylor polynomial is called a Maclaurin polynomial. Taylor and Maclaurin polynomials approximate functions in the sense described above.

Example 1

Find the 4th Maclaurin polynomial for $f(x) = e^x$.

The Maclaurin polynomial is the Taylor polynomial with $c = 0$.

$$\begin{aligned} f(x) &= e^x, & f(0) &= 1 \\ f'(x) &= e^x, & f'(0) &= 1 \\ f''(x) &= e^x, & f''(0) &= 1 \\ f'''(x) &= e^x, & f'''(0) &= 1 \\ f^{(4)}(x) &= e^x, & f^{(4)}(0) &= 1 \end{aligned}$$

It follows that

$$P_4(x) = 1 + (x-0) + \frac{1}{2!}(x-0)^2 + \frac{1}{3!}(x-0)^3 + \frac{1}{4!}(x-0)^4$$

or

$$P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4. \qquad \diamondsuit$$

Example 2

Find the 4th Taylor polynomial for $f(x) = \ln x$ at $x = 1$. Sketch the graph of f and P_4.

$$\begin{aligned} f(x) &= \ln x, & f(1) &= 0 \\ f'(x) &= \frac{1}{x}, & f'(1) &= 1 \\ f''(x) &= -\frac{1}{x^2}, & f''(1) &= -1 \\ f'''(x) &= \frac{2}{x^3}, & f'''(1) &= 2 \\ f^{(4)}(x) &= -\frac{6}{x^4}, & f^{(4)}(1) &= 6 \end{aligned}$$

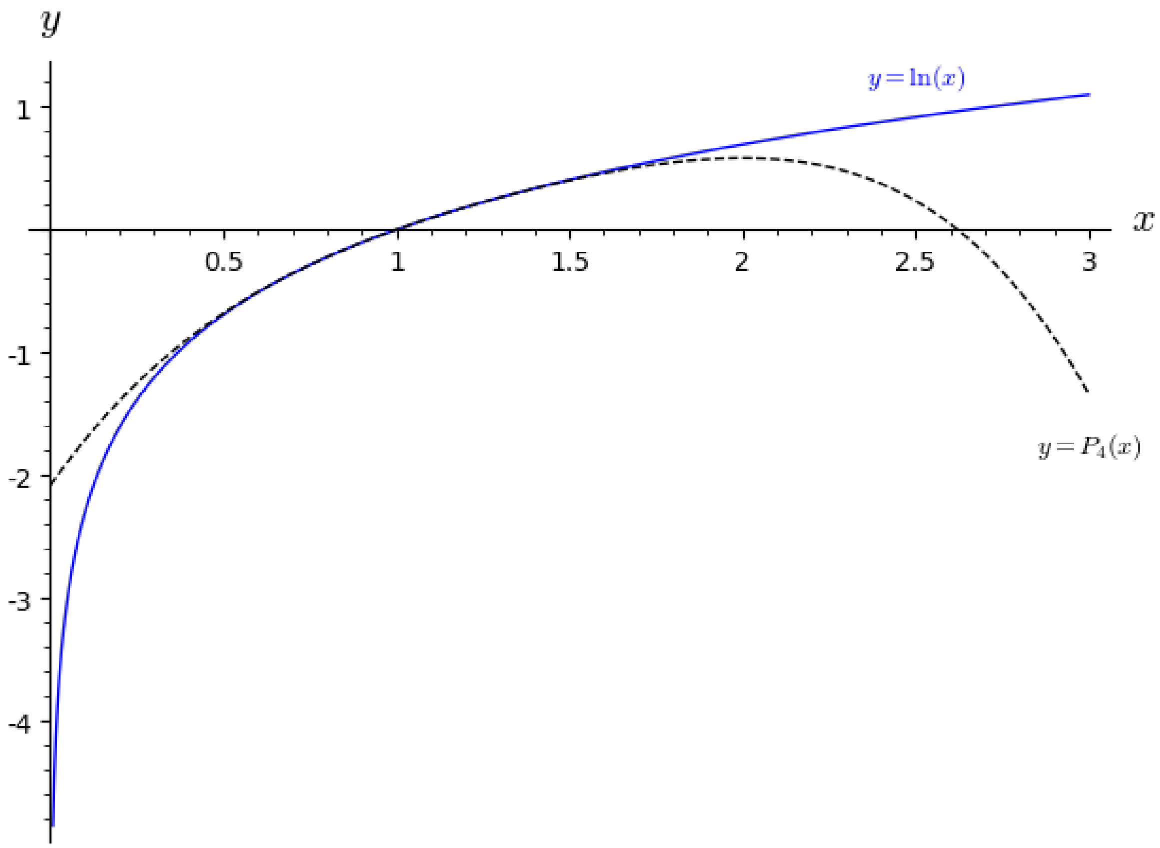
It follows that

$$P_4(x) = 0 + 1(x-1) - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 - \frac{6}{4!}(x-1)^4$$

or

$$P_4(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4.$$

The graphs of $y = \ln(x)$ and $y = P_4(x)$ are shown below. $\diamondsuit$



Definition 2

Suppose f has derivatives of all orders at $x = c$, then the Taylor series for f at $x = c$ is the infinite series given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots.$$

If $c = 0$, the Taylor series is called a Maclaurin series.

Example 3

Find the Maclaurin series for $f(x) = e^x$.

Let's look back at example 1 and generalize from the pattern:

$$1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots + \frac{1}{n!}x^n + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \qquad \diamond$$

Example 4

Find the Taylor series for $f(x) = \ln x$ at $x = 1$.

Once again, let's look at example 2 and generalize from the pattern. We will find that the series is

$$(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n.$$

Now take a minute to look back at example 5 in the lecture notes for section 6.1. In that example, if you replace every x by $x-1$, you'll obtain precisely the Taylor series above. That means that the power series centered at $x=1$ is the same as the Taylor series at $x=1$. This is not a coincidence. $\diamondsuit$

Theorem 1

Suppose f is represented by a power series centered at $x=c$. Also suppose that the power series converges to f on an open interval containing c. Then the power series centered at $x=c$ is identical to the Taylor at $x=c$.

Comments

1. Based on the theorem, we can find the power series for a function by finding the Taylor series for the function. That is awesome! We now have a straight-forward procedure for finding the power series representation for function.
2. The power series representation for a function (which can be obtained from the Taylor series) does not necessarily converge to the function. This is a very important point, and it may sound rather odd. So far we have not seen anything that guarantees that a power series for a function actually converges to that function! We will come back to that later.

Example 5

Find a power series representation centered at $x=0$ for $f(x)=\sin x$.

We find the Maclaurin series for $f(x)=\sin x$...

$$\begin{aligned} f(x) &= \sin x, & f(0) &= 0 \\ f'(x) &= \cos x, & f'(0) &= 1 \\ f''(x) &= -\sin x, & f''(0) &= 0 \\ f'''(x) &= -\cos x, & f'''(0) &= -1 \\ &\vdots \end{aligned}$$

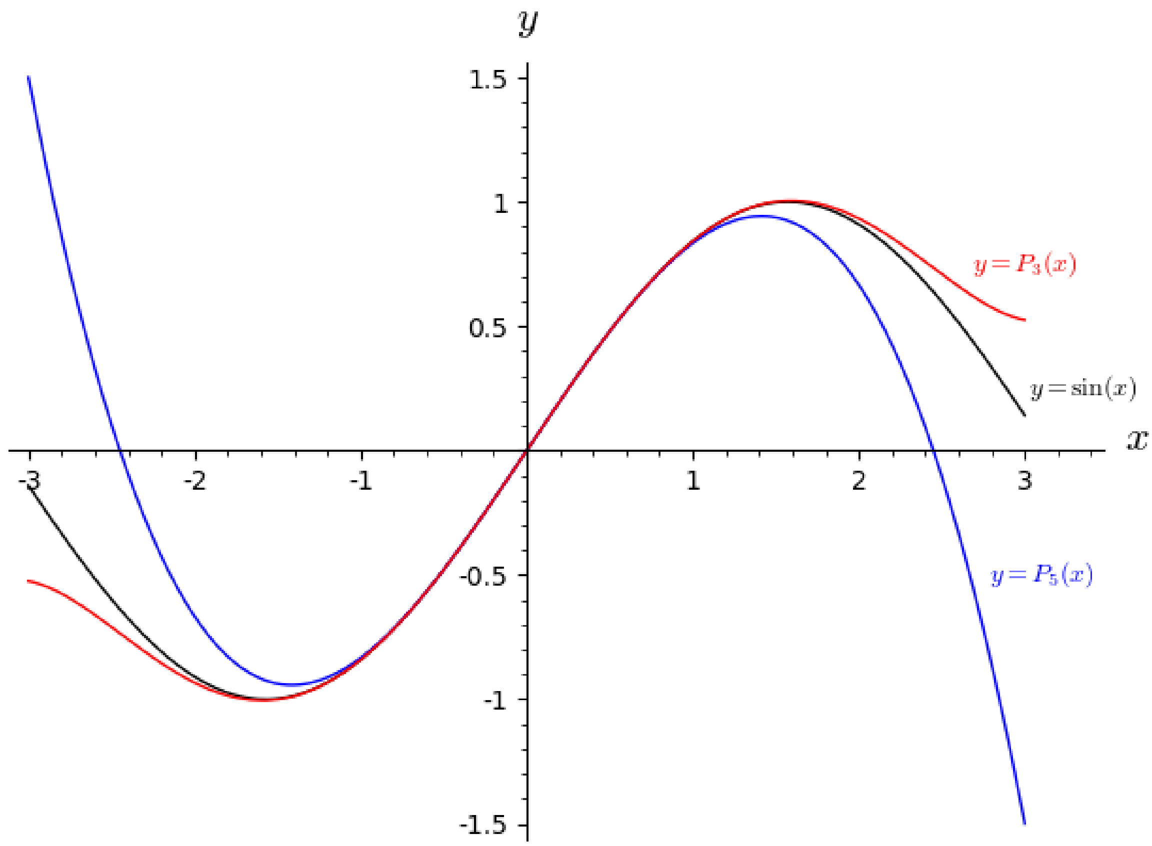
Continuing, we will find that all even-order derivatives at $x=0$ are zero, and the odd-order derivatives at $x=0$ alternate from 1 to -1. So our Maclaurin series is

$$0+1(x-0)+\frac{0}{2!}(x-0)^2-\frac{1}{3!}(x-3)^3+\frac{0}{4!}(x-0)^4+\frac{1}{5!}(x-1)^5+\frac{0}{6!}(x-0)^6-\frac{1}{7!}(x-1)^7+\cdots,$$

and upon cleaning this up, we get

$$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\frac{x^{11}}{11!}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}. \qquad \diamondsuit$$

Shown below are the graphs of $y=\sin x$ and some of the partial sums of its Maclaurin series.



Theorem 2 (Taylor's theorem)

Suppose f is differentiable through order $n+1$ on an interval I containing c. Let $P_n(x)$ be the n th Taylor polynomial for f at $x=c$. Then for each x in I, there exists a number z between x and c such that

$$f(x)=P_n(x)+R_n(x),$$

where

$$R_n(x)=\frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}.$$

$R_n(x)$ is called the nth remainder.

Example 6

Suppose you use the 4th Maclaurin polynomial for e^x to approximate $e^{0.1}$. Use Taylor's theorem to find an upper bound on the error in your approximation.

For $f(x)=e^x$, the 4th Maclaurin polynomial is $P_4(x)=1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}$ (see example 1). According to Taylor's theorem (with $c=0$ and $n=4$), there is a number z between 0 and 0.1 such that

$$e^{0.1}=P_4(0.1)+\frac{f^{(5)}(z)}{120}(0.1)^5.$$

Now, $f^{(5)}(x)=e^x$ so that $f^{(5)}(z)=e^z$. Furthermore, since e^x is an increasing function, it follows that

$$e^{0.1} - P_4(0.1) = \frac{e^z}{120}(0.1)^5 \le \frac{e^{0.1}}{120}(0.1)^5.$$

The upper bound on the right *could be* computed by using a calculator, but stop and think about that! This problem is all about approximating $e^{0.1}$. If we use our calculator to compute it, it defeats the purpose of the problem. Do you understand?

Rather than compute $e^{0.1}$ in order to arrive at a bound, let's just use some information we already know:

$$e^{0.1} < e \approx 2.71828 < 3.$$

Therefore the upper bound satisfies

$$\frac{e^{0.1}}{120}(0.1)^5 < \frac{3}{120}(0.1)^5 = 2.5 \times 10^{-7}.$$

That means the difference between $e^{0.1}$ and $P_4(0.1) = 1.1051708\overline{3}$ is smaller than 0.00000025. That's a pretty good approximation! ♢

Example 7

Suppose you use the 3rd Maclaurin polynomial for $\sin x$ to approximate $\sin(0.125)$. Use Taylor's theorem to find an upper bound on the error in your approximation.

Using Taylor's theorem and looking back to example 5, we have

$$\sin(0.125) = P_3(0.125) + \frac{-\cos z}{4!}(0.125)^4,$$

for some number z between 0 and 0.125.

Since $-1 \le \cos z \le 1$, it follows that

$$|\sin(0.125) - P_3(0.125)| \le \frac{1}{4!}(0.125)^4 \approx 1.017252 \times 10^{-5}.$$

So the difference between $\sin(0.125)$ and $P_3(0.125) = (0.125) - (0.125)^3/6 \approx 0.1246744$ is less than 0.000011. ♢

Theorem 3 (Convergence of Taylor series)

Suppose f has derivatives of all orders on an interval I containing c. Then the Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

converges to $f(x)$ for all x in I if and only if

$$\lim_{n\to\infty} R_n(x) = 0$$

for all x in I.

Example 8

Find the Maclaurin series for $f(x) = \cos x$. Then prove that the Maclaurin series converges to the function at any value of x.

We find the Maclaurin series for $f(x) = \cos x$...

$$\begin{aligned} f(x) &= \cos x, & f(0) &= 1 \\ f'(x) &= -\sin x, & f'(0) &= 0 \\ f''(x) &= -\cos x, & f''(0) &= -1 \\ f'''(x) &= \sin x, & f'''(0) &= 0 \\ &\vdots \end{aligned}$$

Continuing, we will find that all odd-order derivatives at $x = 0$ are zero, and the even-order derivatives at $x = 0$ alternate from 1 to -1. So our Maclaurin series is

$$1 + 0(x-0) - \frac{1}{2!}(x-0)^2 + \frac{0}{3!}(x-3)^3 + \frac{1}{4!}(x-0)^4 + \frac{0}{5!}(x-1)^5 - 1\frac{1}{6!}(x-0)^6 + \frac{0}{7!}(x-1)^7 + \cdots,$$

and upon cleaning this up, we get

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

Next, in order to prove convergence to $f(x) = \cos x$, we must show that the nth remainder approaches zero as n approaches infinity.

$R_n(x)$ has the form

$$R_n(x) = \frac{f^{(n)}(z)}{(n+1)!}\, x^{n+1},$$

and since the nth derivative is a sine or a cosine, its value is between -1 and 1. Therefore

$$|R_n(x)| = \frac{|f^{(n)}(z)|}{(n+1)!}\, |x|^{n+1} \le \frac{|x|^{n+1}}{(n+1)!}.$$

It is probably not obvious that the expression on the right approaches zero as $n \to \infty$. Here is a clever way to see it.

Look back at example 4 in the lecture notes for section 6.1. In that example, we used the ratio test to show that the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges absolutely for all x. By the nth term test for divergence, it follows that the terms of the series approach zero. So we must have

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0.$$

Our final conclusion is that for any number x,

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots. \qquad \diamondsuit$$

Example 9

Functions do not necessarily converge to their series representations. Here is a classic example due to Cauchy.

Consider the function

$$f(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \ne 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Although we will not work through the details here, f is infinitely differentiable, and it has a Maclaurin series that converges for all real numbers. In fact, its Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, x^n = \sum_{n=0}^{\infty} 0\, x^n = 0.$$

Its Maclaurin series is identically zero! On the other hand, $f(x)$ is nonzero everywhere except at $x = 0$. Even though both are defined everywhere, $f(x)$ is equal to its Maclaurin series only at one point. ♢