Lecture 10: Introduction to tangent lines

Objectives:

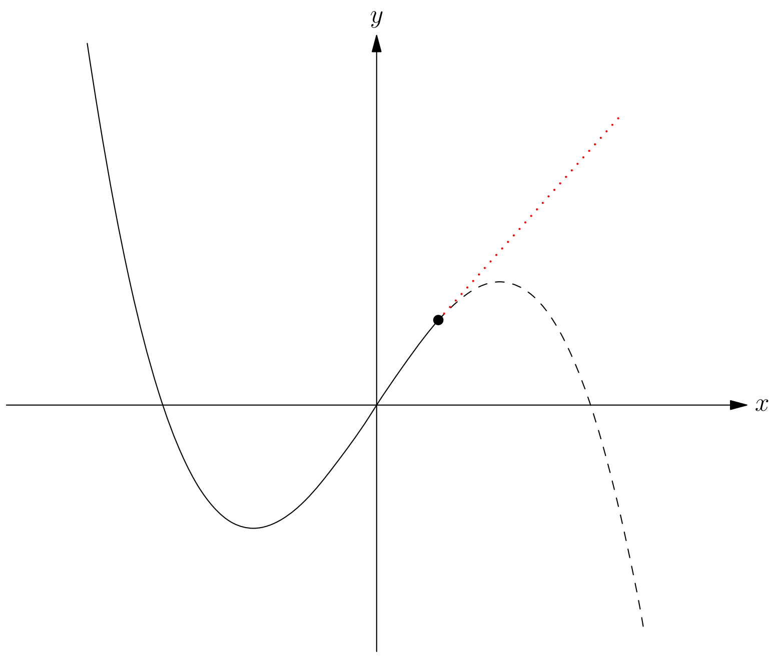
(10.1) Interpret the derivative as the slope of a tangent line.

(10.2) Use the graph of a function to determine properties of its derivative.

Trajectories and tangent lines

Suppose that a moving particle is constrained to a continuous curve (like a train on a track), but there are no other forces acting on it. What would happen to the particle if the constraint force suddenly disappeared? According to Newton's First Law: *Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.*

In the figure below, if a particle is moving to the right along the curve, but becomes unconstrained at the indicated point, then it will follow the dotted trajectory. That dotted trajectory is the "right line" to which Newton referred. Such a line is called a tangent line.



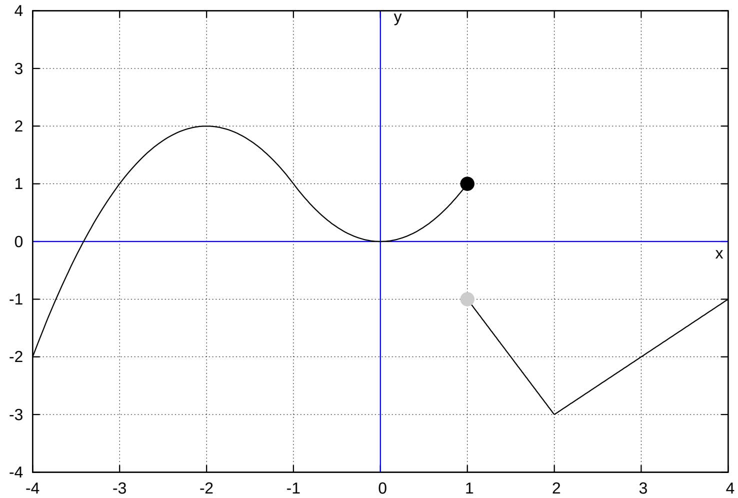
Regardless of the direction of the particle's motion, if the constraint force disappeared, the particle would continue along a tangent line that is determined by the curve at the point at which the force ceased.

> **Informal Definition of Derivative**
> If the graph of a function has a unique tangent line at a point (regardless of direction), then the function is said to be differentiable at that point. Furthermore, the slope of the tangent line at that point is called the derivative at the point.

Example 1 Refer to the graph above. At the indicated point, is the derivative of the function positive, negative, or zero?

Since the slope of the tangent line is positive, the derivative is positive.

Example 2 Consider the graph of $y = f(x)$ shown below.



1. At which point(s) is f not differentiable?

   Particles moving to the left and right at $x = 1$ and $x = 2$ would follow different trajectories. Unique tangent lines do not exist at those points. f is not differentiable at $x = 1$ and $x = 2$.

2. At which points does the derivative of f have positive values?

   Tangent lines have positive slopes at all points in the open intervals $(-4, -2)$, $(0, 1)$, and $(2, 4)$.

3. At which points does the derivative of f have negative values?

   Tangent lines have negative slopes at all points in the open intervals $(-2, 0)$ and $(1, 2)$.

4. At which points is the derivative of f equal to zero?

   Tangent lines are horizontal at $x = -2$ and $x = 0$.

At this point we have only an informal, graphical conception of the derivative. Nonetheless, some very important ideas should be clear. First of all, we certainly cannot expect a function to be differentiable at a point where it is not continuous. If we rephrase this idea, we get the following theorem.

Theorem 1 — Differentiability implies continuity
If f is differentiable at $x = c$, then f is continuous at $x = c$.

We should be careful not to misinterpret this theorem. Differentiable functions are continuous, but continuous functions are not necessarily differentiable. For example, $g(x) = |x|$ is continuous everywhere, but the graph's sharp point at the origin shows that g is not differentiable at $x = 0$. In fact, there exist functions that are continuous everywhere, but differentiable nowhere.

Another important idea concerns increasing and decreasing functions. A function f is said to be increasing on an interval I if the values of f are consistently greater as we move to the right in I. A similar definition describes what it means to be decreasing on I. The following theorem should almost be obvious.

Theorem 2 — Increasing/decreasing functions
If f is differentiable at each point of (a, b) and the derivative is positive at each point, then f is increasing on (a, b).

If f is differentiable at each point of (a, b) and the derivative is negative at each point, then f is decreasing on (a, b).

In order to prove theorems 1 and 2, we'll need a more rigorous definition of the derivative. But they are easy enough to believe, and we will not hesitate to use them.

Example 3 Sketch the graph of a function that is continuous everywhere, but not differentiable at $x = 0$, $x = 2$, and $x = 5$.

Answers vary. Solution omitted.

Example 4 Sketch the graph of $y = (x - 3)^2 + 1$ and then determine intervals on which the derivative is positive/negative.

Solution omitted.

Example 5 Sketch the graph of a function whose derivative is positive on $(-2, 4)$ and negative on $(-\infty, -2) \cup (4, \infty)$.

Answers vary. Solution omitted.