Lecture 11: The limit definition of the derivative

Objectives:

(11.1) State, explain, and illustrate the limit definition of derivative.

(11.2) Use the limit definition of derivative to evaluate a derivative.

(11.3) Find equations of tangent lines.

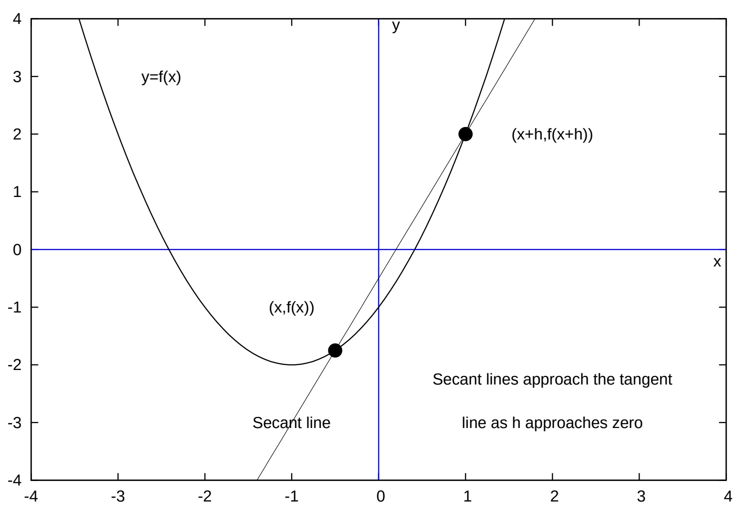
(11.4) Determine when a derivative does not exist.

Formal definition of derivative

We have already discussed the idea of the derivative as the slope of a tangent line. It turns out the derivative concept is incredibly useful, but right now we can only estimate derivatives graphically by drawing tangent lines and approximating slopes. We need an approach by which we can find a derivative exactly and analytically. The approach we will take is simple (in theory) yet amazingly clever. To compute the derivative of f at an arbitrary point x:

1. The coordinates of the point on the graph of f are $(x,\ f(x))$.
2. Think about a small change in x, either to the left or right, and call it h (or Δx). The x-value at the new location is $x + h$.
3. The coordinates of the point on the graph of f at the new location are $(x + h,\ f(x + h))$.
4. Find the slope of the secant line passing through the two points by computing rise over run.
5. Take the limit as $h \to 0$.

A secant line is a line through two points on a graph. In the procedure outlined above, we are simply thinking of the tangent line as a limit of secant lines.



Formal Limit Definition of Derivative
The derivative of f at x is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

provided this limit exists. If $f'(x)$ exists, we say f is differentiable at x. The process of finding the derivative of a function is called differentiation.

(Direct substitution of $h = 0$ into the definition will **always** yield the indeterminate form $0/0$. There will always be more work to do.)

Notice that we not only have an analytic definition of derivative, but we also have notation for it. There are many ways to denote the derivative of $y = f(x)$. Here are some of the most common:

$$f'(x), \qquad y', \qquad \frac{dy}{dx}, \qquad \frac{d}{dx}[f(x)]$$

Example 1 Use the definition to determine the derivative of $f(x) = 3x + 5$.

$$\begin{aligned} f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{[3(x+h)+5] - [3x+5]}{h} \\ = \lim_{h\to 0} \frac{3x + 3h + 5 - 3x - 5}{h} &= \lim_{h\to 0} \frac{3h}{h} = 3 \end{aligned}$$

Notice that f' is a constant function. Regardless of where we look, the derivative is always 3. This should be pretty obvious since the graph of f is a line, and the derivative measures slope.

Example 2 Use the definition to evaluate $\frac{d}{dx}(x^2 + x)$.

$$\begin{aligned} \frac{d}{dx}(x^2+x) &= \lim_{h\to 0} \frac{[(x+h)^2 + (x+h)] - [x^2+x]}{h} = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 + x + h - x^2 - x}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2 + h}{h} = \lim_{h\to 0} \frac{h(2x+h+1)}{h} = \lim_{h\to 0} (2x+h+1) = 2x+1 \end{aligned}$$

If $f(x) = x^2 + x$, then $f'(x) = 2x + 1$.

Example 3 Let $f(x) = x^2 + x$. Using the result from above, find an equation of the line tangent to the graph of f at the point where $x = 1$.

The point on the graph is $(1, f(1))$ or $(1, 2)$. The slope at this point is given by $f'(1) = 2(1)+1 = 3$. The tangent line is given by $y - 2 = 3(x - 1)$ or $y = 3x - 1$.

Example 4 Let $g(x) = \sqrt{x+5}$.

1. Use the definition to compute $g'(x)$.

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{\sqrt{(x+h)+5} - \sqrt{x+5}}{h} \cdot \frac{\sqrt{(x+h)+5} + \sqrt{x+5}}{\sqrt{(x+h)+5} + \sqrt{x+5}} = \lim_{h\to 0} \frac{[x+h+5] - [x+5]}{h(\sqrt{x+h+5} + \sqrt{x+5})} \\ &= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h+5} + \sqrt{x+5})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h+5} + \sqrt{x+5}} = \frac{1}{2\sqrt{x+5}} \end{aligned}$$

2. What is the slope of the line tangent to the graph of g at the point where $x = 4$.

$$g'(4) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

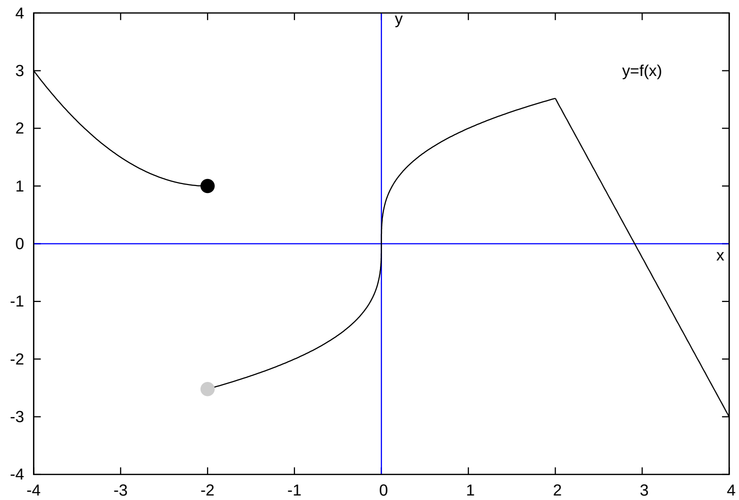
3. Find an equation of the tangent line at $x = 4$.

The point is $(4, g(4)) = (4, 3)$. The slope is $g'(4) = 1/6$. The tangent line is given by $y - 3 = \frac{1}{6}(x - 4)$.

Points at which a derivative fails to exist

Along with our informal graphical approach to the derivative, which we saw in Lecture 10, we found that the derivative of a function cannot exist at a point of discontinuity or at a point at which the graph makes a sharp change. Since the derivative measures slope, and the slope of a vertical line is not defined, a derivative also cannot exist where a graph has a vertical tangent line.

Example 5 The graph of f is shown below. At which points does f' not exist?



f is not differentiable at $x = -2$ because f is discontinuous there. f is not differentiable at $x = 0$ because the tangent line is vertical. f is not differentiable at $x = 2$ because there are different tangent lines from the left and right.

Common Ways Derivatives Fail to Exist

f is not differentiable at $x = c$ if

- f is discontinuous at $x = c$,
- the slope of the graph's tangent line from the left at $x = c$ is not equal to the slope of the tangent line from the right (i.e. the graph has a sharp point at $x = c$), or
- the tangent line at $x = c$ is vertical.