Lecture 12: Some basic differentiation rules

Objectives:

(12.1) Use basic differentiation rules to evaluate derivatives.

Basic differentiation rules

Just as the limit laws helped us simplify the process of finding limits, there are differentiation rules that simplify the process of finding derivatives. We will use these rules on a daily basis instead of using the formal definition of derivative. The formal definition will be reserved for proofs and special problems. The first two rules are easy to believe if we remember that derivatives measure slope.

> If c, m, and b are any real numbers, then
>
> $$\frac{d}{dx}[c] = 0 \qquad \text{and} \qquad \frac{d}{dx}[mx+b] = m.$$

Example 1 Find the derivative of each function.

1. $f(x) = -3$

   f is a constant function. Its graph is a horizontal line. The derivative is the constant function $f'(x) = 0$.

2. $g(x) = 8x$

   g is a linear function. Its graph is a line with slope 8. The derivative is the constant function $g'(x) = 8$.

3. $s(t) = 7 - 5t$

   s is a linear function. Its graph is a line with slope -5. The derivative is the constant function $s'(t) = -5$.

The next rule is rather difficult to prove. We'll leave the proof for later (or never).

> **The Power Rule**
> If n is any real number and $f(x) = x^n$, then
>
> $$f'(x) = nx^{n-1},$$
>
> provided both f and f' are defined at x.

Example 2 Evaluate each derivative.

1. $\frac{d}{dx}x^5$ Solution: $\frac{d}{dx}x^5 = 5x^4$
2. $\frac{d}{dt}\frac{1}{t^2}$ Solution: $\frac{d}{dt}\frac{1}{t^2} = \frac{d}{dt}t^{-2} = -2t^{-3} = -\frac{2}{t^3}$
3. $\frac{d}{dw}\sqrt[3]{w^2}$ Solution: $\frac{d}{dw}\sqrt[3]{w^2} = \frac{d}{dw}w^{2/3} = \frac{2}{3}w^{-1/3} = \frac{2}{3\sqrt[3]{w}}$

Sum, Difference, and Constant Multiple Rules
If c is any real number and f and g are differentiable functions, then

- $\dfrac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$
- $\dfrac{d}{dx}[f(x)-g(x)] = f'(x)-g'(x)$
- $\dfrac{d}{dx}[cf(x)] = c\,\dfrac{d}{dx}[f(x)] = cf'(x)$

Example 3 Evaluate each derivative.

1. $\dfrac{d}{dx}(5x)^2$ Solution: $\dfrac{d}{dx}(5x)^2 = \dfrac{d}{dx}[25x^2] = 25\,\dfrac{d}{dx}x^2 = (25)(2x) = 50x$
2. $\dfrac{d}{dx}(5x^7+\sqrt{x})$ Solution: $\dfrac{d}{dx}(5x^7+\sqrt{x}) = 5\,\dfrac{d}{dx}x^7 + \dfrac{d}{dx}x^{1/2} = (5)(7x^6) + \dfrac{1}{2}x^{-1/2} = 35x^6 + \dfrac{1}{2\sqrt{x}}$
3. $\dfrac{d}{dx}[4x^3-7x^2+8x-9]$

   Solution: $\dfrac{d}{dx}[4x^3-7x^2+8x-9] = 4\,\dfrac{d}{dx}x^3 - 7\,\dfrac{d}{dx}x^2 + \dfrac{d}{dx}(8x) - \dfrac{d}{dx}(9) = 12x^2-14x+8$

Sine and Cosine Rules

$$\frac{d}{dx}[\sin x] = \cos x \qquad\qquad \frac{d}{dx}[\cos x] = -\sin x$$

Proof of the derivative rule for the sine function

$$\begin{aligned}
\frac{d}{dx}[\sin x] &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x\cos h - \sin x}{h} + \lim_{h\to 0}\frac{\cos x\sin h}{h} \\
&= \sin x\left(\lim_{h\to 0}\frac{\cos h - 1}{h}\right) + \cos x\left(\lim_{h\to 0}\frac{\sin h}{h}\right) \\
&= (\sin x)(0) + (\cos x)(1) = \cos x
\end{aligned}$$

(The proof for the cosine function is similar.)

Example 4 Let $r(\theta) = 5\sin\theta - 7\cos\theta$. Find $r'(\theta)$.

$$r'(\theta) = 5\cos\theta - 7(-\sin\theta) = 5\cos\theta + 7\sin\theta$$

Miscellaneous examples

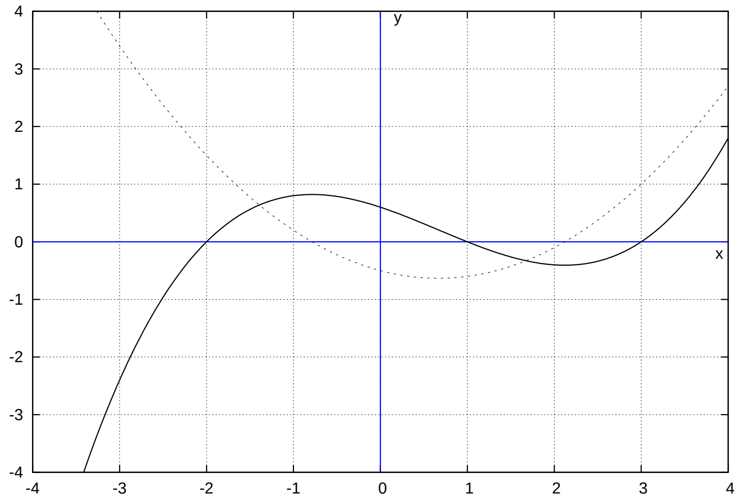
Example 5 Find the slope of the line tangent to the graph of $y = 3x-2+5\sin x$ at the point where $x=\pi$.

The slope is given by dy/dx at $x = \pi$.

$$\frac{dy}{dx} = 3 + 5\cos x$$

The slope is $3 + 5\cos\pi = 3 + 5(-1) = -2$.

Example 6 The graph of a function f and its derivative are shown below.



1. Which is the graph of f, and which is the graph of f'?

   The solid graph is that of f, and the dotted graph is that of f'. Notice that f is increasing where f' is positive, and f is decreasing where f' is negative.

2. Use the graphs to determine $f(3)$ and $f'(3)$.

   $f(3) = 0$ (solid graph) and $f'(3) = 1$ (dotted graph).

3. Find an equation of the line tangent to the graph of f at $x = 3$.

   $y - 0 = 1(x - 3)$ or $y = x - 3$

Example 7 Find the slope of the line tangent to the graph of $g(x) = 9\cos x - \sqrt{5x}$ at the point where $x = \pi$.

We need g', but let's rewrite g so that the derivative is easier to find.

$$g(x) = 9\cos x - \sqrt{5}\,\sqrt{x} = 9\cos x - \sqrt{5}\,x^{1/2}$$

Now we have $g'(x) = -9\sin x - \sqrt{5}(\frac{1}{2})x^{-1/2}$, so that $g'(\pi) = -\sqrt{5}/2 \cdot \pi^{-1/2} \approx -0.63078$.