Lecture 16: Implicit differentiation

Objectives:

(16.1) Use implicit differentiation to find the derivative of an implicitly-defined function.

(16.2) Find equations of the lines tangent and normal to the graph of an implicitly-defined function.

Implicit differentiation

Now that we've mastered the chain rule, we're ready to look at some of its applications. The first such application is implicit differentiation. There is really nothing implicit about the differentiation we'll be doing, but rather there is something implicit about the function being differentiated.

When a function is defined by means of an equation of the form $y = f(x)$, we say that y is explicitly defined as a function of x. For example, here are some equations which explicitly define one variable as a function of the other:

$$y = x^2, \quad y = \sin x, \quad w = x\sin^3 x, \quad s = -16t^2 + 50, \quad z = y^2 + 2^y.$$

When a function is defined by means of an equation in x and y, but the equation is not solved for y, then we say that y is implicitly defined as a function of x. For each of the equations below, we can think of one variable as being implicitly defined as a function of the other.

$$2x + 5y = 7, \quad xy = 1, \quad x^3 + y^3 - 6xy = 0, \quad s^2 \cos t = 7, \quad \frac{y+z}{2z-y} = 1$$

Of course, if we think back to the definition of function and the vertical line test, it's pretty clear that not every equation actually defines a function. For example, the graph of $x^2 + y^2 = 1$ is a circle of radius one centered at the origin. This graph certainly does not pass the vertical line test. However, with suitable restrictions placed on the variables (e.g. $y \geq 0$), the equation $x^2 + y^2 = 1$ indeed defines a function. Although we will normally not worry about the "suitable restrictions", they will always be behind the scenes when we are working with implicit functions.

Our goal right now is to find the derivative of an implicitly-defined function. The procedure is straightforward. Given an equation that implicitly defines y as a function of x, we use the following steps to find dy/dx:

1. Differentiate both sides of the equation with respect to x, treating y as an unknown function of x.
2. Solve the resulting equation for dy/dx.

As a first example, consider the ellipse described by the equation $x^2 + 3y^2 = 18$. In order to find dy/dx, we differentiate, but we must be careful to treat y as a function of x. When we differentiate $3y^2$, we will need the chain rule—the outside function is $3x^2$ and the inside function is y (and the derivative of y is simply dy/dx).

$$\begin{aligned} \frac{d}{dx}[x^2 + 3y^2] &= \frac{d}{dx}[18] \\ \frac{d}{dx}[x^2] + \frac{d}{dx}[3y^2] &= 0 \\ 2x + 6y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} = \frac{-2x}{6y} &= -\frac{x}{3y} \end{aligned}$$

Notice that the final result contains both x's and y's. Although we don't see this when differentiating explicit functions, it is commonplace for implicit functions.

Example 1 Find the slope of the line tangent to the graph of $x^2 + y^2 = 2$ at the point $(-1, 1)$.

We must find dy/dx at $(-1, 1)$.

$$\frac{d}{dx}[x^2 + y^2] = \frac{d}{dx}[2]$$

$$\frac{d}{dx}[x^2] + \frac{d}{dx}[y^2] = 0$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$$

$$\frac{dy}{dx} \text{ at } (-1, 1) = 1$$

The Wolfram Alpha syntax for the implicit differentiation is: `find dy/dx if x^2+y^2=2`.

Example 2 Suppose that y is implicitly defined as a function of x by the equation $x = (x + 1)\sin y^2$. Find dy/dx.

$$\frac{d}{dx}[x] = \frac{d}{dx}[(x + 1)\sin y^2]$$

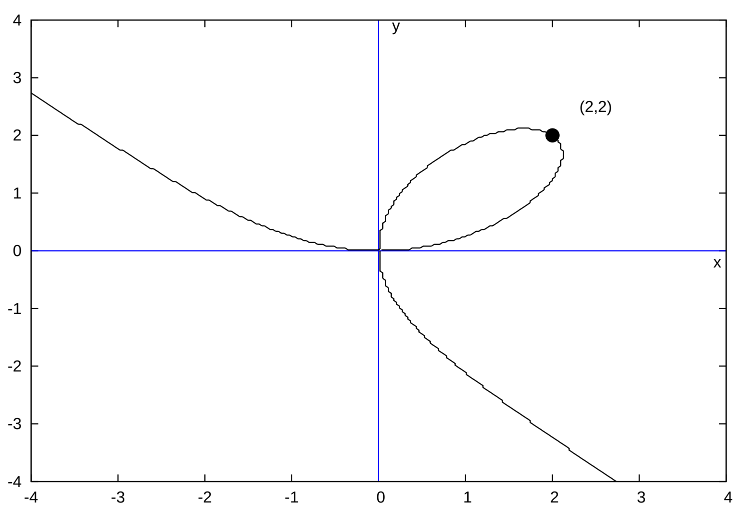
$$1 = (1)(\sin y^2) + (x + 1)(\cos y^2)\left(2y\frac{dy}{dx}\right)$$

$$1 - \sin y^2 = 2y(x + 1)(\cos y^2)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1 - \sin y^2}{2y(x + 1)\cos y^2}$$

The Wolfram Alpha syntax is: `find dy/dx if x=(x+1)*sin(y^2)`.

Example 3 The graph shown below is called a folium of Descartes. It is the graph of the equation $x^3 + y^3 - 4xy = 0$.



1. Find an equation of the line tangent to the graph at the point $(2, 2)$.

   It is easy to verify that $(2, 2)$ actually satisfies the equation. So, we have our point, we just need the slope at $(2, 2)$. The differentiation will require both the chain rule and the product rule.

$$\frac{d}{dx}[x^3 + y^3 - 4xy] = 0$$

$$3x^2 + 3y^2\frac{dy}{dx} - 4y - 4x\frac{dy}{dx} = 0$$

$$(3y^2 - 4x)\frac{dy}{dx} = 4y - 3x^2$$

$$\frac{dy}{dx} = \frac{4y - 3x^2}{3y^2 - 4x}$$

$$\frac{dy}{dx} \text{ at } (2,2) = -1$$

An equation of the tangent line is $y - 2 = -1(x - 2)$ or $y = 4 - x$.

The Wolfram Alpha syntax for the implicit differentiation is: `find dy/dx if x^3+y^3-4*x*y=0`.

2. Find an equation of the normal line passing through $(2, 2)$.

   A normal line is a line perpendicular to a tangent line. The slope of the normal line at $(2, 2)$ is the opposite reciprocal of the slope of the tangent line. Therefore an equation of the normal line is $y - 2 = 1(x - 2)$ or $y = x$.

Example 4 Suppose that y is implicitly defined as a function of x by the equation $y^2 + y = x^2$. Find d^2y/dx^2.

To find the second derivative, we differentiate the first derivative.

$$\frac{d}{dx}[y^2 + y] = \frac{d}{dx}[x^2]$$

$$2y\frac{dy}{dx} + \frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{2x}{2y+1}$$

$$\frac{d^2y}{dx^2} = \frac{(2y+1)(2) - (2x)(2\frac{dy}{dx})}{(2y+1)^2}$$

$$\frac{d^2y}{dx^2} = \frac{(2y+1)(2) - (4x)(\frac{2x}{2y+1})}{(2y+1)^2}$$

After simplifying this expression, we get

$$\frac{d^2y}{dx^2} = \frac{2}{(2y+1)^3}.$$