Lecture 17: Derivatives of inverse functions

Objectives:

(17.1) Compute the derivative of an inverse function.

(17.2) Evaluate derivatives involving the inverse trigonometric functions.

Short review of inverse functions

Two functions are inverses (of one another) if one function undoes the action of the other and vice versa. For example,

$$f(x) = x + 5 \qquad \text{and} \qquad g(x) = x - 5$$

are inverses because each undoes the other. More formally, we have the following definition.

Inverse Functions

The functions f and g are inverse functions if

$$f(g(x)) = x \text{ for all } x \text{ in the domain of } g$$

and

$$g(f(x)) = x \text{ for all } x \text{ in the domain of } f.$$

In this case, we write $g = f^{-1}$ or $f = g^{-1}$.

Example 1 Show that $f(x) = 2x^3 + 1$ and $g(x) = \sqrt[3]{\dfrac{x-1}{2}}$ are inverses.

Let's start with $f(g(x))$.

$$f(g(x)) = 2(g(x))^3 + 1 = 2\left(\sqrt[3]{\frac{x-1}{2}}\right)^3 + 1$$

The right-hand side reduces to x, and it does so for any real number x. Therefore, $f(g(x)) = x$ for all x in the domain of g.

Now let's check $g(f(x))$.

$$g(f(x)) = \sqrt[3]{\frac{f(x)-1}{2}} = \sqrt[3]{\frac{(2x^3+1)-1}{2}}$$

The right-hand side reduces to x, and it does so for any real number x. Therefore, $g(f(x)) = x$ for all x in the domain of f.

A function has an inverse if and only if it is one-to-one. Graphically, one-to-one functions pass the horizontal line test. Inverse functions have a number of important properties.

Properties of inverse functions

Suppose f and f^{-1} are inverse functions. Then

1. f and f^{-1} are one-to-one functions,
2. the domain of f^{-1} is the range of f,
3. the range of f^{-1} is the domain of f,
4. $y = f(x)$ if and only if $f^{-1}(y) = x$,
5. the point (x, y) is on the graph of f if and only if (y, x) is on the graph of f^{-1}, and
6. the graph of f^{-1} is the reflection of the graph of f about the line $y = x$.

Example 2 Let $f(x) = x^3 + 3x + 5$. It may not be obvious, but f is one-to-one on the entire real number line, and therefore f^{-1} exists. Without trying to determine the inverse function, find $f^{-1}(9)$.

$f^{-1}(9)$ must exist. Let's call it w, so that $f^{-1}(9) = w$. Referring to property 4 (above), $f^{-1}(9) = w$ means the same as $f(w) = 9$.

$$f(w) = 9 \Longrightarrow w^3 + 3w + 5 = 9.$$

It is easy to verify that $w = 1$. So $f^{-1}(9) = 1$.

Derivatives of inverse functions

Based on the relationship between the graph of a function and its inverse (property 6), it seems reasonable to believe that "nice" functions have "nice" inverses. This idea is summarized in the following theorem. You should be able to convince yourself of the truth of each part by thinking about the graph of f and its reflection about $y = x$.

Theorem 1 — Continuity and differentiability of inverse functions

Suppose that the function f is defined on the interval I and that f^{-1} exists.

1. If f is continuous on its domain, then f^{-1} is continuous on its domain.
2. If f is increasing (or decreasing) on its domain, then f^{-1} is increasing (or decreasing) on its domain.
3. If f is differentiable and f' is nonzero on an open interval in I, then f^{-1} is differentiable on the corresponding interval in its domain.

Now let's focus our attention on part 3 of theorem 1. We'll assume that the conditions of the theorem hold and, in order to simplify notation, let $g = f^{-1}$. We would like to determine a formula for g'. To that end, notice that

$$f(g(x)) = x,$$

since f and g are inverses. Differentiate both sides of the equation (using the chain rule on the left) to get

$$f'(g(x))\, g'(x) = 1$$

or

$$g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0.$$

We have our formula!

Derivative formula for inverse function

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}, \text{ assuming the denominator is nonzero.}$$

Example 3 Let $f(x) = x^3 + 3x + 5$. This is the same function from example 2.

1. Compute the derivative of f^{-1} at $x = 9$.

   Using the formula above, we have

   $$\left.\frac{d}{dx} f^{-1}(x)\right|_{x=9} = \frac{1}{f'(f^{-1}(9))}.$$

   Since $f'(x) = 3x^2 + 3$ and $f^{-1}(9) = 1$ (from example 2), it follows that

   $$\left.\frac{d}{dx} f^{-1}(x)\right|_{x=9} = \frac{1}{f'(1)} = \frac{1}{6}.$$

2. Find an equation of the line tangent to the graph of $y = f^{-1}(x)$ at the point where $x = 9$.

We know the slope of the tangent line, $m = 1/6$. And, from example 2, we have the point $(9, 1)$. Therefore, the tangent line is described by

$$y - 1 = \frac{1}{6}(x - 9) \qquad \text{or} \qquad y = \frac{1}{6}x - \frac{1}{2}.$$

Example 4 Recall that the inverse of the sine function is denoted by $\arcsin x$ or $\sin^{-1} x$. Find a formula for $\frac{d}{dx}\sin^{-1} x$.

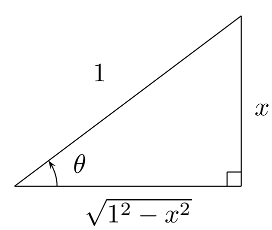
Let $f(x) = \sin x$, so that $f'(x) = \cos x$. Using the formula above,

$$\frac{d}{dx}f^{-1}(x) = \frac{d}{dx}\sin^{-1} x = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\cos(\sin^{-1} x)}.$$

Now, $\cos(\sin^{-1} x)$ can be simplified by using a right triangle. This is a skill you mastered in your trigonometry course. Here is how we do it. Let $\theta = \sin^{-1} x$ and recall (from trig) that $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. We'll sketch θ in the 1st quadrant, but the idea is the same in the 4th quadrant.

Since $\theta = \sin^{-1} x$, it follows that $\sin\theta = \frac{x}{1}$. We interpret $x/1$ as "opposite over hypotenuse" and sketch the corresponding right triangle. After using the Pythagorean theorem to complete the triangle, we can read that

$$\cos\theta = \cos(\sin^{-1} x) = \frac{\text{adj}}{\text{hyp}} = \sqrt{1 - x^2}.$$



It follows that

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}, \quad -1 < x < 1.$$

The formulas for the derivatives of the remaining inverse trigonometric functions can be derived in ways very similar to example 4. The formulas are summarized below.

Derivatives of the inverse trigonometric functions

- $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}, \quad -1 < x < 1$
- $\frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1 - x^2}}, \quad -1 < x < 1$
- $\frac{d}{dx}\tan^{-1} x = \frac{1}{1 + x^2}$
- $\frac{d}{dx}\cot^{-1} x = \frac{-1}{1 + x^2}$
- $\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2 - 1}}, \quad |x| > 1$
- $\frac{d}{dx}\csc^{-1} x = \frac{-1}{|x|\sqrt{x^2 - 1}}, \quad |x| > 1$

For a review of the inverse trigonometric functions, you can find a summary handout at `http://stevekifowit.com/sheets/b04.pdf`.

Example 5 Let $f(x) = x^2 \tan^{-1} x$. Find $f'(x)$.

We will need the product rule and the formula for the derivative of the inverse tangent.

$$f'(x) = \left(\frac{d}{dx}x^2\right)\tan^{-1} x + x^2\left(\frac{d}{dx}\tan^{-1} x\right)$$

$$f'(x) = 2x\tan^{-1} x + \frac{x^2}{1+x^2}$$

The Wolfram Alpha syntax is: `derivative of x^2*arctan(x)`.

Example 6 Let $h(x) = \cos^{-1}(x^4)$. Find $h'(x)$.

Since h is a composition of functions, we will need the chain rule as well as the formula for the derivative of the inverse cosine. The outside function is $f(x) = \cos^{-1} x$, and the inside function is $g(x) = x^4$.

$$\text{Outside: } f(x) = \cos^{-1} x \Longrightarrow f'(x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\text{Inside: } g(x) = x^4 \Longrightarrow g'(x) = 4x^3$$

Putting it all together,

$$h'(x) = \frac{-1}{\sqrt{1-(x^4)^2}}\left(4x^3\right) = \frac{-4x^3}{\sqrt{1-x^8}}$$

The Wolfram Alpha syntax is: `derivative of arccos(x^4)`.