Lecture 18: Derivatives of exponential and logarithmic functions

Objectives:

(18.1) Compute the derivative of an exponential function of any base.

(18.2) Compute the derivative of a logarithmic function of any base.

(18.3) Use logarithmic differentiation.

The natural exponential function

The base-a exponential function is defined by $y = a^x$, where a is a positive real number not equal to 1. Notice that the exponent is variable—these kinds of functions are not power functions! It might be wise to go back to an old algebra book and review the properties of exponential functions. We will not do that here.

Perhaps the most commonly used base is the number e. e can be defined in a number of ways, but we'll use

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \approx 2.71828182845904523536.$$

For now, you will just have to believe that this limit exists[1], but a table of values for increasing n might provide convincing evidence. It turns out that e is irrational, and it is often called the *natural* base.

It is possible to deduce from our definition of e that

$$\lim_{h\to 0}\frac{e^h-1}{h} = 1.$$

In fact, e is the only base with this property. And this limit makes it easy for us to find a formula for the derivative of $f(x) = e^x$.

$$f(x) = e^x \quad\Longrightarrow\quad f'(x) = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^x(e^h-1)}{h} = e^x \lim_{h\to 0}\frac{e^h-1}{h} = e^x$$

Derivative of the natural exponential function

Let $f(x) = e^x$. Then

$$f'(x) = \frac{d}{dx}e^x = e^x$$

and, by the chain rule,

$$\frac{d}{dx}e^{g(x)} = e^{g(x)}\,g'(x).$$

By the way, the notation $\exp(x)$ is also commonly used for the base-e exponential function.

Example 1 Find an equation of the line tangent to the graph of $y = e^x$ at the point where $x = 0$.

When $x = 0$, $y = e^0 = 1$, so we have our point $(0, 1)$. We get the slope from the derivative at $x = 0$.

$$m = \frac{dy}{dx}\bigg|_{x=0} = e^x\bigg|_{x=0} = e^0 = 1.$$

The tangent line is given by

$$y - 1 = 1(x - 0) \qquad\text{or}\qquad y = x + 1.$$

[1] You cannot obtain the limit be direct substitution, because 1^∞ is an indeterminate form.

Example 2 Find the derivative of $y = e^{\cot(5x)}$.

Using the chain rule (twice), we get

$$\frac{dy}{dx} = e^{\cot(5x)}\left[\frac{d}{dx}\cot(5x)\right] = e^{\cot(5x)}(-\csc^2(5x))(5) = -5\cot^2(5x)e^{\cot(5x)}.$$

The Wolfram Alpha syntax is: `derivative of exp(cot(5*x))`.

The natural logarithm

The base-a logarithm, $y = \log_a x$, is defined to be the inverse of the base-a exponential function. That is,

$$y = \log_a x \quad \text{if and only if} \quad a^y = x.$$

As above, it is worth the effort to spend some time reviewing logarithms before proceeding.

The base-e logarithm is the called the natural logarithm, and it is denoted by $\ln x$:

$$\log_e x = \ln x.$$

If we let $f(x) = e^x$, then $f^{-1}(x) = \ln x$, and by using the formula for the derivative of an inverse function (from lecture 17), we have

$$\frac{d}{dx}\ln x = \frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

Derivative of the natural logarithmic function

Let $f(x) = \ln x$. Then

$$f'(x) = \frac{d}{dx}\ln x = \frac{1}{x}$$

and, by the chain rule,

$$\frac{d}{dx}\ln[g(x)] = \frac{1}{g(x)}\,g'(x) = \frac{g'(x)}{g(x)}.$$

Example 3 Let $g(t) = \ln(t^3 + t^2 + t + 1)$. Find $g'(1)$.

$$g'(t) = \frac{1}{t^3 + t^2 + t + 1}\left[\frac{d}{dt}(t^3 + t^2 + t + 1)\right] = \frac{3t^2 + 2t + 1}{t^3 + t^2 + t + 1}.$$

Therefore, $g'(1) = 6/4 = 3/2$.

The Wolfram Alpha syntax is: `derivative of ln(t^3+t^2+t+1) when t=1`.

Example 4 Let $f(x) = \ln\left(\frac{x^2 \sin x}{x^4 + 1}\right)$. Use the properties of logarithms to expand $f(x)$ and then find $f'(x)$.

Using properties of logarithms, we have

$$f(x) = \ln\left(\frac{x^2 \sin x}{x^4 + 1}\right) = 2\ln x + \ln(\sin(x)) - \ln(x^4 + 1).$$

It follows that

$$f'(x) = \frac{2}{x} + \frac{\cos x}{\sin x} - \frac{4x^3}{x^4 + 1}.$$

Logarithmic differentiation

In the last example, we saw that the properties of logarithms can be useful for simplifying derivatives. This idea can actually be used even when a function does not involve a logarithm. If the logarithm properties might be useful, we can simply introduce a logarithm.

For example, suppose we would like to differentiate $y = \dfrac{x^3 \tan x}{e^{2x}}$. Let's use *logarithmic differentiation.*

1. Introduce a log by taking the natural logarithm of each side of the equation:
$$\ln y = \ln\left(\frac{x^3 \tan x}{e^{2x}}\right).$$

2. Use logarithm properties to expand:
$$\ln y = 3\ln x + \ln(\tan x) - 2x.$$

3. Differentiate both sides with respect to x (implicit differentiation on the left!):
$$\frac{y'}{y} = \frac{3}{x} + \frac{\sec^2 x}{\tan x} - 2.$$

4. Solve for y':
$$y' = \frac{dy}{dx} = y\left(\frac{3}{x} + \frac{\sec^2 x}{\tan x} - 2\right) = \frac{x^3 \tan x}{e^{2x}}\left(\frac{3}{x} + \frac{\sec^2 x}{\tan x} - 2\right),$$
where we substituted the original expression back in for y.

Example 5 Let $h(x) = \dfrac{(x+1)(x+2)}{(x+3)(x+4)}$. Use logarithmic differentiation to compute $h'(x)$.

Introduce the logarithm:
$$\ln h(x) = \ln\left(\frac{(x+1)(x+2)}{(x+3)(x+4)}\right) = \ln(x+1) + \ln(x+2) - \ln(x+3) - \ln(x+4).$$

Take the derivative of both sides with respect to x...
$$\frac{h'(x)}{h(x)} = \frac{1}{x+1} + \frac{1}{x+2} - \frac{1}{x+3} - \frac{1}{x+4},$$

and solve for $h'(x)$...
$$h'(x) = \frac{(x+1)(x+2)}{(x+3)(x+4)}\left(\frac{1}{x+1} + \frac{1}{x+2} - \frac{1}{x+3} - \frac{1}{x+4}\right).$$

Bases other than e

Once we get comfortable working with the base-e exponential and logarithm functions, the transition to bases other than e is pretty straight-forward. We will just use the ideas that
$$a^x = e^{\ln a^x} = e^{x \ln a}$$
and
$$\log_a x = \frac{\ln x}{\ln a}.$$
The first fact is simple to verify, and the second is the change-of-base formula you learned in your precalculus course. By using these identities, we can derive the following formulas.

Derivatives when the base is not e

Suppose a is a positive real number not equal to 1. Then

- $\frac{d}{dx}a^x = a^x \ln a$,
- $\frac{d}{dx}a^{g(x)} = a^{g(x)}\, g'(x)\, \ln a$,
- $\frac{d}{dx}\log_a x = \frac{1}{x \ln a}$, and
- $\frac{d}{dx}\log_a g(x) = \frac{g'(x)}{g(x)\ln a}$.

Example 6 Find the slope of the line tangent to the graph of $y = \log_2(3x+1)$ at the point where $x = 1$.

We must compute dy/dx when $x = 1$. To do so, we'll use the derivative formula from above.

$$\frac{dy}{dx} = \frac{3}{(3x+1)(\ln 2)}, \qquad \text{so that } m = \left.\frac{dy}{dx}\right|_{x=1} = \frac{3}{4\ln 2} \approx 1.08202.$$

The Wolfram Alpha syntax is: `derivative of log base 2 of 3*x+1 when x=1`.

Example 7 Let $f(x) = \frac{5^x}{2+5^x}$. Find $f'(x)$.

We will need the quotient rule and the new derivative formula from above.

$$f'(x) = \frac{(5^x)'(2+5^x) - (5^x)(2+5^x)'}{(2+5^x)^2} = \frac{(5^x)(\ln 5)(2+5^x) - (5^x)(5^x)(\ln 5)}{(2+5^x)^2} = \frac{2(5^x)(\ln 5)}{(2+5^x)^2}.$$