Lecture 19: Related rates

Objectives:

(19.1) Use implicit differentiation to relate rates.

(19.2) Solve application problems involving related rates.

Related rate problems

Related rate problems are application problems involving derivatives that are related by the chain rule (or by implicit differentiation). In fact, the alternative form of the chain rule is simply a statement about how rates are related:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Our approach to solving related rate problems will involve four steps:

1. Name variables, list what is given, and list what is to be found. All of this information should be written in math notation.
2. Find one or more equations that relate the variables in our list.
3. Use implicit differentiation to relate rates.
4. Solve for the unknown quantity.

Example 1 A particle is moving along the graph of $y = x^2$ in such a way that $\dfrac{dx}{dt} = 4$. Find $\dfrac{dy}{dt}$ when $x = 3$.

***Step 1*:** There is really nothing to do in this step because everything is spelled out in the problem statement.

***Step 2*:** x and y are related by the equation $y = x^2$.

***Step 3*:** We are interested in derivatives with respect to t, so we differentiate implicitly with respect to t.

$$\frac{d}{dt}[y] = \frac{d}{dt}[x^2]$$

$$\frac{dy}{dt} = 2x\frac{dx}{dt}$$

***Step 4*:** We now solve for dy/dt when $x = 3$.

$$\left.\frac{dy}{dt}\right|_{x=3} = 2 \cdot 3 \cdot 4 = 24$$

Example 2 A large spherical balloon is being inflated at the rate of $5\,\text{ft}^3/\text{min}$. Find the rate of change of the balloon's radius at the moment when the radius is 6 ft.

***Step 1*:** Let t be the elapsed time in minutes, V be the volume of the balloon (ft^3) at time t, and r be the radius (ft) at time t. We are given that $\dfrac{dV}{dt} = 5$, and we need to find $\dfrac{dr}{dt}$ when $r = 6$.

***Step 2*:** V and r are related by the formula for the volume of a sphere: $V = \frac{4}{3}\pi r^3$.

***Step 3*:** We must differentiate with respect to t.

$$\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{4}{3}\pi r^3\right]$$

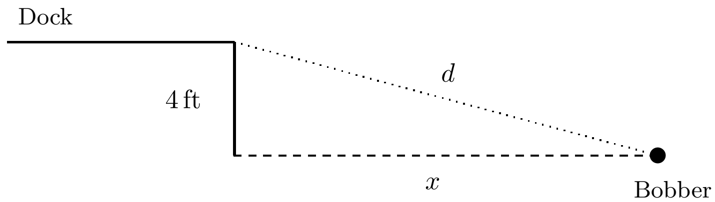
$$\frac{dV}{dt} = 4\pi r^2\,\frac{dr}{dt}$$

***Step 4*:** Use the information above to solve for $\frac{dr}{dt}$ when $r = 6$:

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$

$$\left.\frac{dr}{dt}\right|_{r=6} = \frac{1}{144\pi}\cdot 5 = \frac{5}{144\pi}\,\text{ft/min} \approx 0.01105\,\text{ft/min}$$

Example 3 A fisherman on a dock 4 ft above the surface of the water is reeling in fishing line at a rate of 5 ft/min. How fast is the fisherman's bobber moving along the water surface at the moment when the bobber is 20 ft from the dock?



***Step 1*:** See the diagram above. Let t be the elapsed time (min), x be the distance (ft) from the bobber to the dock, and d be the distance (ft) from the fisherman to the bobber. We are given that $\frac{dd}{dt} = -5$, and we need to find $\frac{dx}{dt}$ when $x = 20$.

***Step 2*:** d and x are related by the Pythagorean theorem: $x^2 + 4^2 = d^2$.

***Step 3*:** We must differentiate with respect to t.

$$\frac{d}{dt}[x^2 + 16] = \frac{d}{dt}[d^2]$$

$$2x\frac{dx}{dt} = 2d\frac{dd}{dt}$$

$$\frac{dx}{dt} = \frac{d}{x}\frac{dd}{dt}$$

***Step 4*:** When $x = 20$, $d = \sqrt{416}$. Therefore,

$$\left.\frac{dx}{dt}\right|_{x=20} = \frac{\sqrt{416}}{20}\cdot(-5) \approx -5.099\,\text{ft}$$

The bobber is moving toward the dock at nearly 5.1 ft/min.

Example 4 The area of a circle is increasing at a rate of 3 ft^2/sec. What is the rate of change of circumference at the moment that the circumference is 24π ft?

***Step 1*:** Let t be the elapsed time (sec), A be the area of the circle (ft^2), C be the circumference (ft), and r be the radius (ft). We are given that $\frac{dA}{dt} = 3$, and we need to find $\frac{dC}{dt}$ when $C = 24\pi$.

***Step 2*:** There are several formulas that relate the variables:

$$A = \pi r^2 \quad \text{and} \quad C = 2\pi r.$$

It follows that $A = \pi\left(\frac{C}{2\pi}\right)^2 = \frac{C^2}{4\pi}$.

***Step 3*:** We must differentiate with respect to t.

$$\frac{d}{dt}[A] = \frac{d}{dt}\left[\frac{C^2}{4\pi}\right]$$

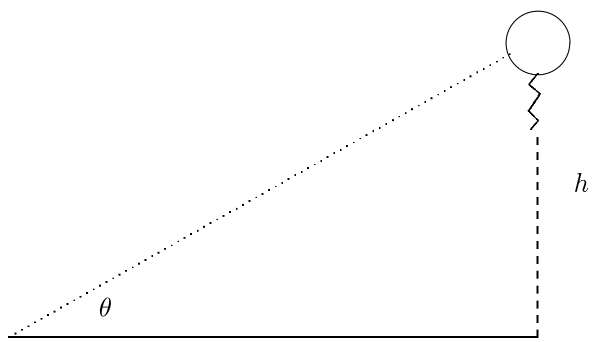
$$\frac{dA}{dt} = \frac{2C}{4\pi}\frac{dC}{dt}$$

$$\frac{dC}{dt} = \frac{4\pi}{2C}\frac{dA}{dt}$$

***Step 4*:** We solve for $\frac{dC}{dt}$ when $C = 24\pi$:

$$\left.\frac{dC}{dt}\right|_{C=24\pi} = \frac{4\pi}{48\pi}\cdot(3) = \frac{1}{4}\text{ ft/sec}$$

Example 5 A balloon is rising straight up from a level field at a rate of 2 m/sec. An observer 50 m downrange is tracking the balloon's ascent. How fast is the angle of elevation from the observer to the balloon changing at the moment when the height is 15 m?



50 m

***Step 1*:** Let t be the elapsed time (sec), h be the height (m) of the balloon at time t, and θ be the angle of elevation from observer to balloon. We are given that $\frac{dh}{dt} = 2$, and we need to find $\frac{d\theta}{dt}$ when $h = 15$.

***Step 2*:** h and θ are related by the equation $\tan\theta = \frac{h}{50}$.

***Step 3*:** We must differentiate with respect to t.

$$\frac{d}{dt}[\tan\theta] = \frac{d}{dt}\frac{h}{50}$$

$$(\sec^2\theta)\frac{d\theta}{dt} = \frac{1}{50}\frac{dh}{dt}$$

***Step 4*:** At the moment when $h = 15$, $\sec\theta = \frac{\sqrt{2725}}{50}$. Therefore, $\sec^2\theta = \frac{2725}{2500}$, and we have

$$\left.\frac{d\theta}{dt}\right|_{h=15} = \frac{2500}{2725}\cdot\frac{1}{50}\cdot(2) \approx 0.0367\text{ radians/sec.}$$