Lecture 21: Differentials

Objectives:

(21.1) Compute differentials.

(21.2) Use differentials to approximate change and to estimate propagated error.

Differentials

If $y = f(x)$, the limit definition of the derivative simply says

$$f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

What's troubling about this definition is that even though we have always been able to treat Δy and Δx as separate things, we have not been able to think about dy and dx separately. Our goal in this lecture is to define dy and dx as separate objects, but ensure that their ratio continues to be the familiar derivative. We will see that this idea has some useful applications.

Definition of differentials

Suppose $y = f(x)$ is a differentiable function. The differential dy is defined by

$$dy = f'(x)\, dx,$$

where the differential dx is simply an independent variable representing a nonzero real number.

Example 1 Compute each differential.

1. dy if $y = \sin 2x$

$$dy = \left(\frac{dy}{dx}\right) dx = 2\cos(2x)\, dx$$

2. dw if $w = \frac{1}{x^3}\, dx$

$$dw = \left(\frac{dw}{dx}\right) dx = -\frac{3}{x^4}\, dx$$

3. dr if $r = \tan^5 t$

$$dr = \left(\frac{dr}{dt}\right) dt = 5\tan^4 t \sec^2 t\, dt$$

Now that we have a way of thinking about dy and dx separately, the following approximation seems reasonable:

$$f'(x) = \frac{dy}{dx} \approx \frac{\Delta y}{\Delta x} \text{ for small } \Delta x$$

or

$$\Delta y \approx dy = f'(x)\, dx \approx f'(x)\, \Delta x.$$

According to the approximation, a derivative multiplied by a small change in x is roughly equal to the corresponding change in y. This provides a quick way to approximate change.

Approximation by differentials

Suppose $y = f(x)$ is a differentiable function. For small Δx,

$$\Delta y = f(x + \Delta x) - f(x) \approx f'(x)\, \Delta x.$$

Example 2 Let $f(x) = \sqrt{x}$. Use differentials to approximate Δy when $x = 4$ and $\Delta x = 0.1$.

The differential dy satisfies $dy = \frac{1}{2\sqrt{x}}\,dx$. Therefore, when $x = 4$ and $\Delta x = 0.1$, we have

$$\Delta y \approx \frac{1}{4}(0.1) = 0.025.$$

Notice that the exact change is $\Delta y = \sqrt{4.1} - \sqrt{4} = 0.024845673131658$.

Example 3 Use differentials to approximate the change in $f(x) = x^3 - 7x^2 + x$ that accompanies the change from $x = 2$ to $x = 2.1$.

By definition, the differential dy satisfies $dy = (3x^2 - 14x + 1)\,dx$. Using the approximation above, we have

$$\Delta y \approx (3x^2 - 14x + 1)\,\Delta x.$$

When x changes from 2 to 2.1, we have $\Delta x = 0.1$. Therefore, when $x = 2$ and $\Delta x = 0.1$,

$$\Delta y \approx (12 - 28 + 1)(0.1) = -1.5.$$

Notice that the exact change is $\Delta y = f(2.1) - f(2) = -1.509$, so our approximation is quite good.

Example 4 Use differentials to approximate $(1.99)^3$.

Let $f(x) = x^3$ and let $\Delta y = f(1.99) - f(2)$. It follows that $\Delta x = -0.01$ and $f'(x) = 3x^2$. Using differentials, we have

$$\Delta y \approx f'(2)\,(-0.01) = -0.12$$

and

$$f(1.99) \approx -0.12 + f(2) = 7.88.$$

Error propagation

When the volume of a sphere is determined by measuring the radius of the sphere, any errors associated with the measurement of the radius affect the computed volume. The error in the volume is called *propagated error*. Differentials can be used to estimate propagated error.

Example 5 The radius of a sphere is measured to be 8 in with an error of no more than 0.15 in. Determine the volume of the sphere and estimate the propagated error.

The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$. It follows that

$$dV = 4\pi r^2\,dr.$$

Using $r = 8$ and $\Delta r = 0.15$, we have

$$\Delta V \approx 4\pi(8)^2(0.15) = 120.6371579.$$

The volume is $2144.66\,\text{in}^3 \pm 120.64\,\text{in}^3$.

Example 6 An object is falling in such a way that its height in meters after t seconds is given by $s(t) = -4.9t^2 + 8t + 10$. Use differentials to estimate the propagated error in $s(t)$ if t is measured to within 0.1 sec.

$\Delta s \approx (-9.8t + 8)(0.1) = -0.98t + 0.8$, where Δs is measured in meters.