Lecture 23: Rolle's Theorem and the Mean Value Theorem

Objectives:

(23.1) State, explain, and apply Rolle's Theorem.

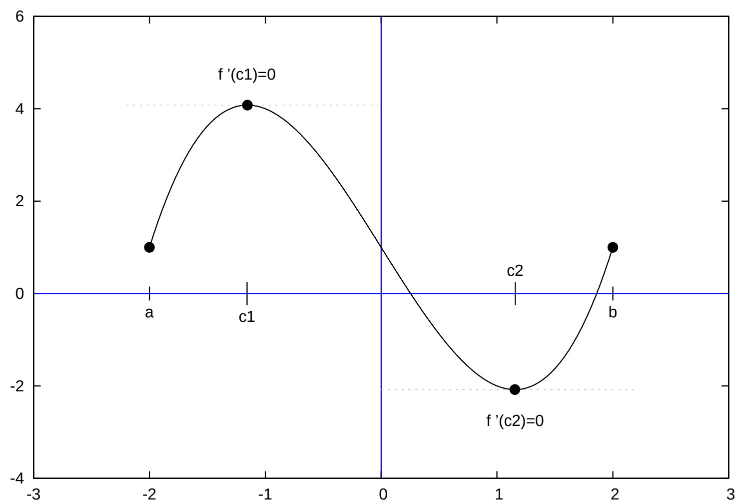
(23.2) State, explain, and apply the Mean Value Theorem.

Rolle's Theorem

In this lecture we will study two theoretical results that will be behind the scenes of some of the most important theorems of calculus. We begin with Rolle's Theorem, named for the French mathematician Michel Rolle (1652-1719).

> **Rolle's Theorem**
>
> Suppose f is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists a number c in (a, b) such that $f'(c) = 0$.

A proof of Rolle's Theorem will not be given here. However, the theorem is easy to believe if we think about it graphically. The requirement that $f(a) = f(b)$ means that the graph of f starts and ends at the same level. The theorem simply states that the graph of a smooth, continuous curve that starts and ends at the same height must have a least one horizontal tangent line.



Example 1 Find a number c that satisfies the conclusion of Rolle's Theorem for the function $f(x) = x^2 - 2x + 2$ on the interval $[-1, 3]$.

f is continuous on $[-1, 3]$ and differentiable on $(-1, 3)$. Furthermore, $f(-1) = 5$ and $f(3) = 5$. Rolle's Theorem applies, and it tells us that $f'(x)$ must be zero somewhere between $x = -1$ and $x = 3$.

$$f'(x) = 2x - 2 = 0 \Longrightarrow x = 1$$

The conclusion of Rolle's Theorem is satisfied by the number $c = 1$.

Example 2 Explain why Rolle's Theorem cannot be applied to the function $f(x) = |x|$ on $[-1, 1]$.

f is continuous on $[-1, 1]$ and $f(-1) = f(1) = 1$. However, f is not differentiable on $(-1, 1)$. In particular, $f'(0)$ does not exist.

Example 3 Use Rolle's Theorem to prove that for any polynomial p, p' has a real zero somewhere between any two distinct real zeros of p.

The proof is a straight-forward application of Rolle's Theorem. Let a and b be two distinct real zeros of the polynomial p. Since p is a polynomial, it must be continuous on $[a, b]$ and differentiable on (a, b). Furthermore, since a and b are zeros of p, we must have $p(a) = p(b) = 0$. It follows immediately from Rolle's Theorem, that there exists a number c in (a, b) such that $p'(c) = 0$. That is, p' has a zero between a and b.

Mean Value Theorem

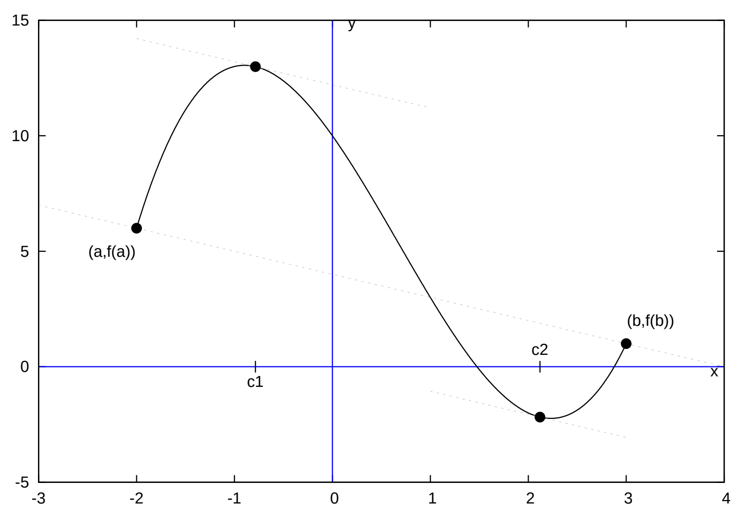
Our next theorem is a generalization of Rolle's Theorem called the Mean Value Theorem. As we will see, the word "mean" refers to an average.

Mean Value Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

A rigorous proof of the Mean Value Theorem (MVT) would make use of Rolle's Theorem. We'll forgo such a proof and stick to a geometric interpretation. In order to illustrate the MVT, we simply look to the illustration of Rolle's Theorem, and keeping the axes fixed, we tilt the graph and the tangent lines. We would get a figure that looks something like this:



According to the MVT, the slope of the secant line through the points $(a, f(a))$ and $(b, f(b))$ must be equal to the slope of the tangent line somewhere between a and b.

Example 4 Find all numbers that satisfy the conclusion of the MVT for $g(x) = x^3 + 3x^2$ on $[-4, 2]$.

Since g is a polynomial, it is certainly continuous on $[-4, 2]$ and differentiable $(-4, 2)$. According to the MVT, there exists a number c such that

$$g'(c) = \frac{g(2) - g(-4)}{2 + 4} = \frac{36}{6} = 6.$$

We now solve $g'(x) = 3x^2 + 6x = 6$ to get $x = -1 - \sqrt{3} \approx -2.732$ or $x = -1 + \sqrt{3} \approx 0.732$. Both lie within the interval $(-4, 2)$, so we have two solutions

$$c_1 = -1 - \sqrt{3} \qquad \text{or} \qquad c_2 = -1 + \sqrt{3}.$$

Another important interpretation of the MVT is: *If f continuous and differentiable on an interval, then the average rate of change over the interval is equal to the instantaneous rate of change at some point in the interval.*

Example 5 The Ohio Turnpike is about 241 mi long and has a speed limit of 65 mph[1]. A car enters the Ohio Turnpike at 8am and reaches the end at 11:20am. As the car exits, the driver is stopped and given a speeding ticket. Use the MVT to explain.

> The driver made the trip in 3 hrs 20 min or roughly 3.33 hrs. The car's average speed was about $241/3.33 \approx 72.4$ mph. According to the MVT, the average speed must be equal to instantaneous speed at some point. Therefore the driver must have exceeded the speed limit somewhere on the turnpike.

As our last example of the MVT, we will prove an important result concerning increasing/decreasing functions.

Definition of increasing/decreasing

Suppose the function f is defined on an interval I.

- If for any two points x_1 and x_2 in I,
$$x_1 < x_2 \quad \Longrightarrow \quad f(x_1) < f(x_2),$$
then f is increasing on I.
- If for any two points x_1 and x_2 in I,
$$x_1 < x_2 \quad \Longrightarrow \quad f(x_1) > f(x_2),$$
then f is decreasing on I.

We first encountered the following theorem in lecture 10 during our informal discussion of the derivative. We are now prepared to prove it.

Theorem 1 — Increasing/decreasing functions

If f is differentiable at each point of (a, b) and the derivative is positive at each point, then f is increasing on (a, b).

If f is differentiable at each point of (a, b) and the derivative is negative at each point, then f is decreasing on (a, b).

Proof: We will only prove the first part of the theorem. A proof of the second part is similar.

Suppose f is differentiable at each point of (a, b). Let x_1 and x_2 be any two points in (a, b) with the property that $x_1 < x_2$. Since f is differentiable on (a, b), it must be continuous on (a, b). Therefore f must be continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). According to the MVT, there exists a number c in (x_1, x_2) such that
$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1},$$
or
$$f(x_2) - f(x_1) = f'(c) \cdot (x_2 - x_1).$$
By hypothesis, $f'(c)$ must be positive, and since $x_2 > x_1$, the right hand side of the equation above must be positive. Therefore,
$$f(x_2) - f(x_1) > 0 \qquad \text{or} \qquad f(x_2) > f(x_1).$$
It follows that f is increasing on (a, b).

[1]When these notes were first written, this was true. Now the speed limit is greater in most areas.