Lecture 24: First derivative test

Objectives:

(24.1) Use the first derivative to find intervals on which a function is increasing/decreasing.

(24.2) Use the first derivative test to locate relative extrema.

Increasing/decreasing functions

We have already discussed the following theorem, but it is worth stating again.

Theorem 1 — Increasing/decreasing functions

If f is differentiable at each point of (a, b) and the derivative is positive at each point, then f is increasing on (a, b).

If f is differentiable at each point of (a, b) and the derivative is negative at each point, then f is decreasing on (a, b).

Now suppose that f is a continuous function on the interval (a, b). Also suppose that f is differentiable on (a, b), except possibly at some isolated points. If the sign of f' changes, say from positive to negative, then f makes a corresponding change from increasing to decreasing. We are interested in tracking these changes, but to do so, we must first think about how the derivative can change sign. Given that f is continuous and mostly differentiable, f' can only change sign in one of two ways:

1. by passing through a point at which $f'(x) = 0$ or
2. by passing over a point at which $f'(x)$ does not exist.

Theorem 2 — First derivative test for relative extrema

Suppose f is continuous on the interval (a, b). Also suppose that f is differentiable on (a, b), except possibly at some isolated points.

- If $f'(x)$ changes from negative to positive at a critical number c, then $f(c)$ is a relative minimum.
- If $f'(x)$ changes from positive to negative at a critical number c, then $f(c)$ is a relative maximum.
- If $f'(x)$ has the same sign on both sides of a critical number c, then $f(c)$ is neither a relative minimum nor a relative maximum.

Finding intervals on which f is increasing/decreasing

Suppose that f is continuous on the interval I. Also suppose that f is differentiable inside I, except possibly at some isolated points. We use the following steps to find the relative extrema and open intervals on which f is increasing/decreasing.

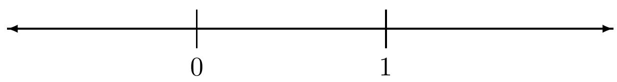
1. Determine domain endpoints (if any) and all points at which $f'(x)$ is zero or not defined.
2. Draw a number line and mark the points found in step 1. (Indicate which are critical numbers.)
3. Determine the sign of f' on each interval along your number line.
4. List open intervals on which f is increasing/decreasing.
5. Identify and list all relative extrema, including both x- and y-coordinates.

Example 1 Find open intervals on which $f(x) = 2x^3 - 3x^2 + 1$ is increasing/decreasing. Also identify all relative extreme values.

f is defined everywhere—there are no domain endpoints.

$$f'(x) = 6x^2 - 6x = 0 \Longrightarrow 6x(x-1) = 0 \Longrightarrow x = 0 \text{ or } x = 1$$

There are no x-values for which $f'(x)$ DNE. We must only mark $x = 0$ and $x = 1$ on our number line. Both are critical numbers.

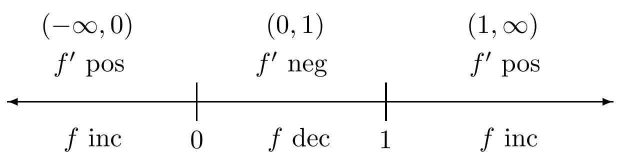


We now determine the sign of f' on each interval. Recall that $f'(x) = 6x(x-1)$.

$$x \text{ in } (-\infty, 0) \Longrightarrow f'(x) > 0$$

$$x \text{ in } (0, 1) \Longrightarrow f'(x) < 0$$

$$x \text{ in } (1, \infty) \Longrightarrow f'(x) > 0$$



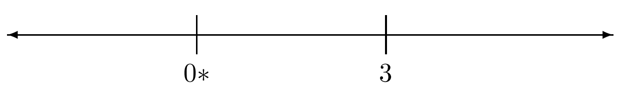
Final answer: f is increasing on $(-\infty, 0) \cup (1, \infty)$ and decreasing on $(0, 1)$. $f(0) = 1$ is a relative maximum and $f(1) = 0$ is a relative minimum.

Example 2 Find open intervals on which $f(x) = \dfrac{2x-3}{x^2}$ is increasing/decreasing. Also identify all relative extreme values.

f is defined everywhere except $x = 0$.

$$f'(x) = \frac{(x^2)(2) - (2x-3)(2x)}{x^4} = \frac{-2x^2 + 6x}{x^4} = \frac{6 - 2x}{x^3}$$

$f'(x) = 0$ only when $x = 3$, and $f'(x)$ DNE when $x = 0$. We must mark $x = 0$ and $x = 3$ on our number line, but $x = 0$ is not a critical number (since it is not in the domain of f).

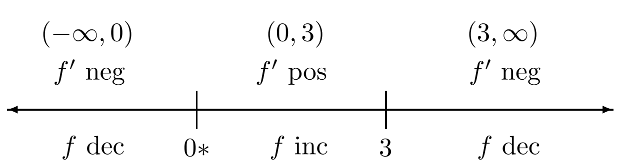


We now determine the sign of f' on each interval. Recall that $f'(x) = \dfrac{2(3-x)}{x^3}$.

$$x \text{ in } (-\infty, 0) \Longrightarrow f'(x) < 0$$

$$x \text{ in } (0, 3) \Longrightarrow f'(x) > 0$$

$$x \text{ in } (3, \infty) \Longrightarrow f'(x) < 0$$



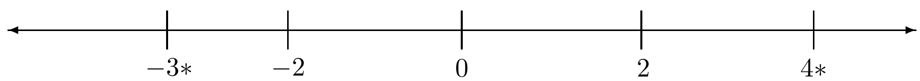
Final answer: f is decreasing on $(-\infty, 0) \cup (3, \infty)$ and increasing on $(0, 3)$. $f(3) = 1/3$ is a relative maximum.

Example 3 Let $g(x) = x^4 - 8x^2 + 3$ on $[-3, 4]$. Find open intervals on which g is increasing/decreasing. Identify all relative extreme values. Also find the absolute extreme values.

g is defined on $[-3, 4]$. The domain endpoints are $x = -3$ and $x = 4$.

$$g'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x-2)(x+2) = 0 \implies x = 0, x = 2, \text{ or } x = -2$$

There are no x-values for which $g'(x)$ DNE. We must mark $x = -3, -2, 0, 2, 4$ on our number line. The critical numbers are $x = -2, 0, 2$. $x = -3$ and $x = 4$ are not critical numbers.



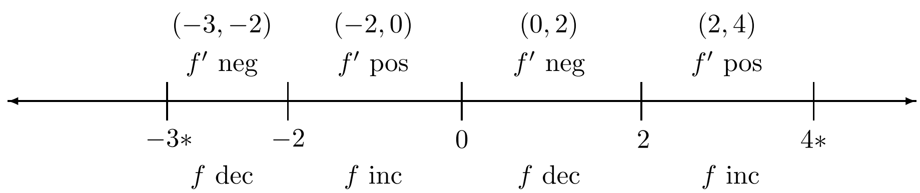
We now determine the sign of g' on each interval. Recall that $g'(x) = 4x(x-2)(x+2)$.

$$x \text{ in } (-3, -2) \implies f'(x) < 0$$

$$x \text{ in } (-2, 0) \implies f'(x) > 0$$

$$x \text{ in } (0, 2) \implies f'(x) < 0$$

$$x \text{ in } (2, 4) \implies f'(x) > 0$$



We can use the sign chart above to determine the relative extrema, but we need a table of function values to determine the absolute extrema.

x	-3	-2	0	2	4
$g(x)$	12	-13	3	-13	131

Final answer: g is decreasing on $(-3, -2) \cup (0, 2)$ and increasing on $(-2, 0) \cup (2, 4)$. $g(-2) = -13$ and $g(2) = -13$ are relative minima. $g(0) = 3$ is a relative maximum. The absolute maximum is $g(4) = 131$, and the absolute minimum is $g(-2) = g(2) = -13$.