Lecture 25: Second derivative test

Objectives:

(25.1) Use the second derivative to find intervals on which the graph of a function is concave up/down.

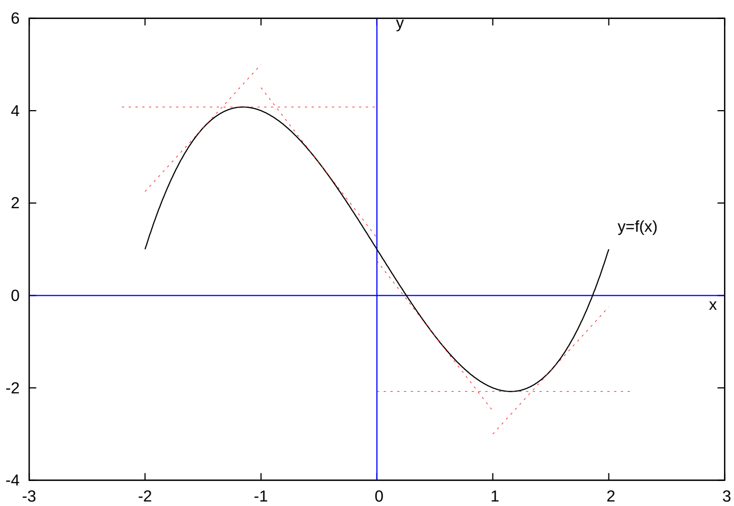
(25.2) Find the points of inflection of the graph of a function.

(25.3) Apply the second derivative test to classify relative extrema.

Concavity and the 2nd derivative

Now that we have studied the first derivative test, it is natural to wonder if there is such a thing as a second derivative test. By analogy with the first derivative test, we would expect that a second derivative test would tell us what the sign of the 2nd derivative says about the original function. Let's take a look.

The figure below shows the graph of a function f, along with several of its tangent lines.



Notice that the slopes of the tangent lines are decreasing until somewhere near $x = 0$. Once past the y-axis, the slopes are increasing. When the slopes are decreasing (i.e. f' is decreasing), f'' must be negative, and when the slopes are increasing (i.e. f' is increasing), f'' must be positive. In the figure above when the slopes are decreasing, the graph bends down or is **concave down**. When the slopes are increasing, the graph bends up or is **concave up**. A shape that is concave down "spills water," whereas a shape that is concave up "holds water."

Definition of concavity

Suppose f is differentiable on (a, b).

- If f' is increasing on (a, b), then the graph of f is concave up on (a, b).
- If f' is decreasing on (a, b), then the graph of f is concave down on (a, b).

With this definition of concavity, the next result follows immediately from theorem 1 of lecture 24.

Theorem 1 — Test for concavity

Suppose f is a function such that f'' exists on (a, b).

- If $f''(x) > 0$ for all x in (a, b), then the graph of f is concave up on (a, b).
- If $f''(x) < 0$ for all x in (a, b), then the graph of f is concave down on (a, b).

In order to determine intervals on which a graph is concave up/down, we must be able to identify points at which the concavity might change. For reasons that will be clear later, we will call these points *possible inflection points* or PIP's for short. (The PIP's are also sometimes referred to as Hergert numbers[1].)

Definition of PIP (or Hergert number)

An interior point in the domain of f at which $f''(x) = 0$ or $f''(x)$ does not exist is called a PIP or a Hergert number.

PIP's (Hergert numbers) are domain points at which a graph's concavity might change.

Example 1 Find the PIP's for $f(x) = \dfrac{5}{x^2+1}$. Does the graph's concavity actually change at each one?

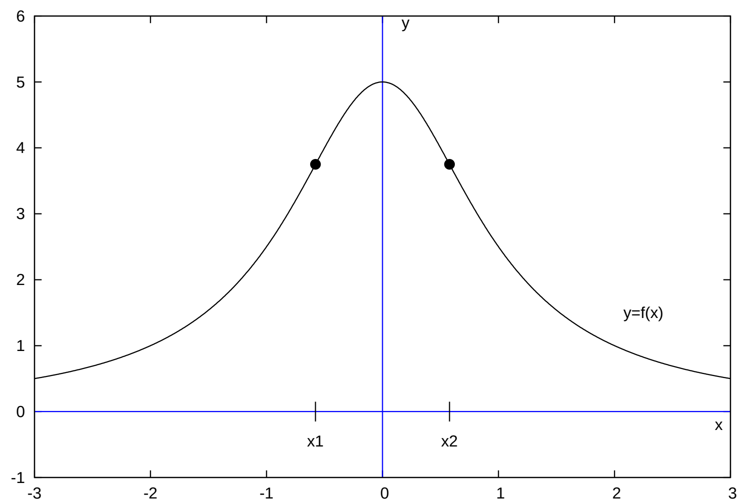
$$f'(x) = \frac{(x^2+1)(0) - (5)(2x)}{(x^2+1)^2} = \frac{-10x}{(x^2+1)^2}$$

$$f''(x) = \frac{(x^2+1)^2(-10) - (-10x)(2)(x^2+1)(2x)}{(x^2+1)^4} = \frac{30x^2 - 10}{(x^2+1)^3}$$

$f''(x)$ is defined everywhere, and $f''(x) = 0$ when $x = 1/\sqrt{3}$ or $x = -1/\sqrt{3}$. The PIP's of f are

$$x_1 = -\frac{1}{\sqrt{3}} \quad \text{and} \quad x_2 = \frac{1}{\sqrt{3}}.$$

To the left of x_1, $f''(x) > 0$. Between x_1 and x_2, $f''(x) < 0$. And to the right of x_2, $f''(x) > 0$. The concavity actually changes at each PIP. The graph of f is shown below.



In the example above, the concavity of the graph changed at each of the PIP's. When the concavity changes like this, the corresponding point on the graph is called an **inflection point**.

Definition of inflection point

Suppose that f is continuous on the interval (a, b) containing the point c. If the concavity of the graph of f changes at the point c, then $(c, f(c))$ is called a point of inflection of the graph of f.

Warning: Authors disagree about the best definition of inflection point. You will see a lack of consistency when referring to different sources.

[1] Hergert numbers (http://www.hergertnumbers.org) are named for Rodger Hergert, a mathematics professor at Rock Valley College (Rockford, IL).

In order to locate inflection points analytically, we must look for the PIP's. numbers.

Theorem 2 — Inflection points and PIP's

If $(c, f(c))$ is a point of inflection of the graph of f, then c is a PIP (or Hergert number) for f.

Finding intervals on which the graph of f is concave up/down

Suppose that f is continuous on the interval I. Also suppose that f is twice differentiable inside I, except possibly at some isolated points. We use the following steps to find open intervals on which the graph of f is concave up/down.

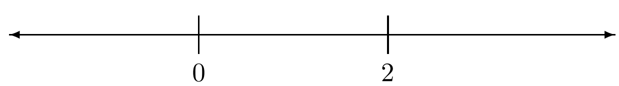
1. Determine domain endpoints (if any) and all points at which $f''(x)$ is zero or not defined. (This list will include all PIP's.)
2. Draw a number line and mark the points found in step 1.
3. Determine the sign of f'' on each interval along your number line.
4. List open intervals on which the graph of f is concave up/down.
5. Identify all inflection points.

Example 2 Determine open intervals on which the graph of $g(x) = x^4 - 4x^3$ is concave up/down. Also identify all inflection points of the graph of g.

Since g is a polynomial, it is defined everywhere and has derivatives of all orders.

$$g''(x) = 12x^2 - 24x = 12x(x-2) = 0 \quad \Longrightarrow \quad x = 0 \text{ or } x = 2$$

The PIP's are $x = 0$ and $x = 2$, and these numbers must be marked on our number line.

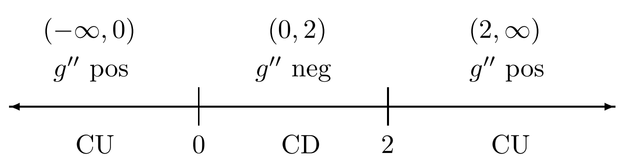


We now determine the sign of g'' on each interval. Recall that $g''(x) = 12x(x-2)$.

$$x \text{ in } (-\infty, 0) \Longrightarrow g''(x) > 0$$

$$x \text{ in } (0, 2) \Longrightarrow g''(x) < 0$$

$$x \text{ in } (2, \infty) \Longrightarrow g''(x) > 0$$



Final answer: The graph of g is concave up on $(-\infty, 0) \cup (2, \infty)$ and concave down on $(0, 2)$. g is continuous at $x = 0$ and $x = 2$, and the concavity changes at these points. Therefore $(0, 0)$ and $(2, -16)$ are points of inflection.

Second derivative test

We conclude lecture 25 with a simple observation.

Theorem 3 — Second derivative test for relative extrema

Suppose that f is twice differentiable on an open interval containing c and that $f'(c) = 0$.

- If $f''(c) > 0$, then the graph of f is concave up at c and $f(c)$ is a relative minimum.
- If $f''(c) < 0$, then the graph of f is concave down at c and $f(c)$ is a relative maximum.
- If $f''(c) = 0$, then this test fails to be useful and another test must be applied.

Example 3 Use the 2nd derivative test to show that $f(x) = 3x^4 - 16x^3 + 18x^2$ has a relative maximum at $x = 1$.

$$f'(x) = 12x^3 - 48x^2 + 36x = 12x(x-3)(x-1) = 0 \quad \Longrightarrow \quad x = 0, x = 1, \text{ or } x = 3$$
$$f''(x) = 36x^2 - 96x + 36$$
$$f''(1) = -24$$

Since $f''(1) < 0$, $f(1) = 5$ is a relative maximum.