Lecture 26: Limits at infinity

Objectives:

(26.1) Evaluate limits at infinity.

(26.2) Find the horizontal asymptotes of the graph of a function.

Limits at infinity

In this lecture we'll take a break from applications of derivatives and return to the study of limits. In particular we will study limits at infinity and their applications to curve sketching.

In an earlier lecture, we learned to assign infinite limits to functions that were growing without bound as a limit point was approached. We will now consider what may happen to a function when the limit point itself grows without bound. For example, as x grows, the values of $1/x$ get smaller and smaller. In fact, the bigger x gets, the closer $1/x$ gets to zero. In the formal language of limits, $1/x$ can be made arbitrarily close to zero by choosing x sufficiently large. We write

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

It is also true that

$$\lim_{x\to-\infty} \frac{1}{x} = 0.$$

In order to evaluate limits at infinity, we can use the two limits listed above as well as our earlier limit laws. For the most part, those limit laws continue to hold true for limits at infinity (either $+\infty$ or $-\infty$). For example, as long as all of the individual limits exist, we have

- $\lim\limits_{x\to\pm\infty} [f(x) + g(x)] = \lim\limits_{x\to\pm\infty} f(x) + \lim\limits_{x\to\pm\infty} g(x)$,
- $\lim\limits_{x\to\pm\infty} k \cdot f(x) = k \cdot \lim\limits_{x\to\pm\infty} f(x)$, or
- $\lim\limits_{x\to\pm\infty} [f(x)g(x)] = \lim\limits_{x\to\pm\infty} f(x) \cdot \lim\limits_{x\to\pm\infty} g(x)$.

Our other limit laws also hold when applicable.

Example 1 Evaluate each limit.

1. $\lim\limits_{x\to\infty} \left(2 + \dfrac{3}{x}\right)$

$$\lim_{x\to\infty} \left(2 + \frac{3}{x}\right) = \lim_{x\to\infty} 2 + 3 \lim_{x\to\infty} \frac{1}{x} = 2 + 3(0) = 2$$

2. $\lim\limits_{x\to-\infty} \dfrac{9}{x^3}$

$$\lim_{x\to-\infty} \frac{9}{x^3} = 9\left(\lim_{x\to-\infty} \frac{1}{x}\right)^3 = 9(0)^3 = 0$$

3. $\lim\limits_{x\to\infty} \dfrac{5 + x^2}{x^2}$

$$\lim_{x\to\infty} \frac{5 + x^2}{x^2} = \lim_{x\to\infty} \left(\frac{5}{x^2} + \frac{x^2}{x^2}\right) = 0 + 1 = 1$$

If we use the basic facts

$$\lim_{x\to\infty} x = \infty \qquad\qquad \lim_{x\to-\infty} x = -\infty$$

along with the interpretation that the form $k/\infty = 0$, then the limits in the previous example could have been evaluated by direct substitution. For example, by using direct substitution,

$$\lim_{x\to\infty}\left(2+\frac{3}{x}\right) \text{ has the form } 2+\frac{3}{\infty}$$

Therefore, the limit is 2. This approach technically would not work for part 3 of the example, since

$$\lim_{x\to\infty}\frac{5+x^2}{x^2} \text{ has the form } \frac{\infty}{\infty}.$$

The form ∞/∞ is an indeterminate form, exactly like the form $0/0$. Other common indeterminate forms involving ∞ are:

$$-\frac{\infty}{\infty},\quad \infty-\infty,\quad 0\cdot\infty,\quad 1^{\infty}.$$

For now, we will mostly be interested in the indeterminate forms $\pm\infty/\infty$. As we did in the example above, we will use algebraic techniques to resolve indeterminate forms.

Resolving ∞/∞

One approach to resolving the form ∞/∞ is to divide through by the highest power of x that appears in the denominator. Then re-evaluate the limit.

In other words, if x^r is the highest power of x in the denominator, then multiply both the numerator and denominator by $\frac{1}{x^r}$.

Example 2 Use the technique described above to evaluate each limit.

1. $\displaystyle\lim_{x\to\infty}\frac{7x^4-8x^2+13}{5x^4+3x^3-2x}$

   The highest power of x in the denominator is x^4.

$$\lim_{x\to\infty}\frac{7x^4-8x^2+13}{5x^4+3x^3-2x}\cdot\frac{\frac{1}{x^4}}{\frac{1}{x^4}}=\lim_{x\to\infty}\frac{7-(8/x^2)+(13/x^4)}{5+(3/x)-(2/x^3)}=\frac{7-0+0}{5+0-0}=\frac{7}{5}$$

2. $\displaystyle\lim_{x\to-\infty}\frac{x^3+7x}{5x^5+4x^4}$

   The highest power of x in the denominator is x^5.

$$\lim_{x\to-\infty}\frac{x^3+7x}{5x^5+4x^4}\cdot\frac{\frac{1}{x^5}}{\frac{1}{x^5}}=\lim_{x\to-\infty}\frac{(1/x^2)+(7/x^4)}{5+(4/x)}=\frac{0+0}{5+0}=0$$

3. $\displaystyle\lim_{x\to\infty}\frac{3x^4-8x^3+2x}{x^2+7x}$

   The highest power of x in the denominator is x^2.

$$\lim_{x\to\infty}\frac{3x^4-8x^3+2x}{x^2+7x}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}}=\lim_{x\to\infty}\frac{3x^2-8x+(2/x)}{1+(7/x)}$$

   If we now "plug in" $x=\infty$, we get the indeterminate form $\infty-\infty$, but this is easy to resolve.

$$\lim_{x\to\infty}(3x^2-8x)=\lim_{x\to\infty}x(3x-8)=\infty\cdot\infty=\infty$$

Example 3 Evaluate each limit.

1. $\displaystyle\lim_{x\to\infty}\frac{8x+5}{\sqrt{4x^2+9}}$

Because the x^2 occurs in the denominator inside the radical, we must say that the highest power of x in the denominator is $\sqrt{x^2}$. How this expression simplifies depends on the sign of x:

$$\sqrt{x^2} = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Since we're taking a limit as $x \to \infty$, we can safely assume $x > 0$.

$$\lim_{x\to\infty} \frac{8x+5}{\sqrt{4x^2+9}} \cdot \frac{\frac{1}{\sqrt{x^2}}}{\frac{1}{\sqrt{x^2}}} = \lim_{x\to\infty} \frac{8x+5}{\sqrt{4x^2+9}} \cdot \frac{\frac{1}{x}}{\frac{1}{\sqrt{x^2}}} = \lim_{x\to\infty} \frac{8+(5/x)}{\sqrt{4+(9/x^2)}} = \frac{8}{2} = 4$$

2. $\displaystyle\lim_{x\to-\infty} \frac{8x+5}{\sqrt{4x^2+9}}$

Referring to the first part of this example, we must divide through by $\sqrt{x^2}$. Since we're taking a limit as $x \to -\infty$, we can safely assume $x < 0$, and therefore $\sqrt{x^2} = -x$.

$$\lim_{x\to-\infty} \frac{8x+5}{\sqrt{4x^2+9}} \cdot \frac{\frac{1}{\sqrt{x^2}}}{\frac{1}{\sqrt{x^2}}} = \lim_{x\to-\infty} \frac{8x+5}{\sqrt{4x^2+9}} \cdot \frac{\frac{1}{-x}}{\frac{1}{\sqrt{x^2}}} = \lim_{x\to-\infty} \frac{-8-(5/x)}{\sqrt{4+(9/x^2)}} = -\frac{8}{2} = -4$$

The Wolfram Alpha syntax for a limit at infinity is: `limit f(x) as x->infinity`. For example, `limit (8*x+5)/sqrt(4*x^2+9) as x->-infinity` produces -4 as expected.

Example 4 Use the squeeze theorem to evaluate the limit: $\displaystyle\lim_{x\to\infty} \frac{\sin x}{x}$

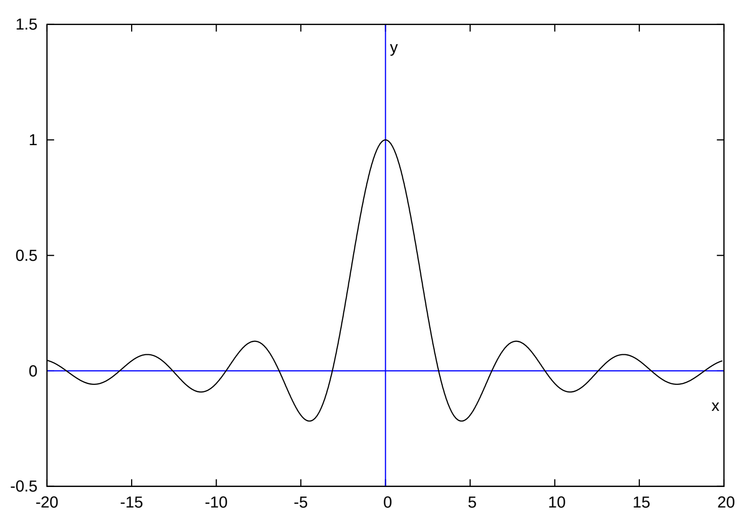
We must be careful not to confuse this limit at infinity with the limit, $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1$. In this case, we use the fact that $\sin x$ is a bounded function whose values are always between -1 and 1. It follows that

$$\frac{\sin x}{x} \text{ is always between } \frac{-1}{x} \text{ and } \frac{1}{x}.$$

Since both $-1/x$ and $1/x$ approach zero as $x \to \infty$, we must have

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0.$$

The function $y = \dfrac{\sin x}{x}$ is important in digital signal processing applications, where it is called sinc(x). The graph is shown below. Even though the graph is continually oscillating, it is getting closer and closer to the x-axis. This is consistent with our observation that the limit at infinity is zero.



Horizontal asymptotes

In the example above, the graph of the function got closer and closer to the x-axis as x got bigger and bigger. This type of behavior is used to define the concept of a *horizontal asymptote*.

Definition of horizontal asymptote

The horizontal line $y = a$ is a horizontal asymptote of the graph of the function f if

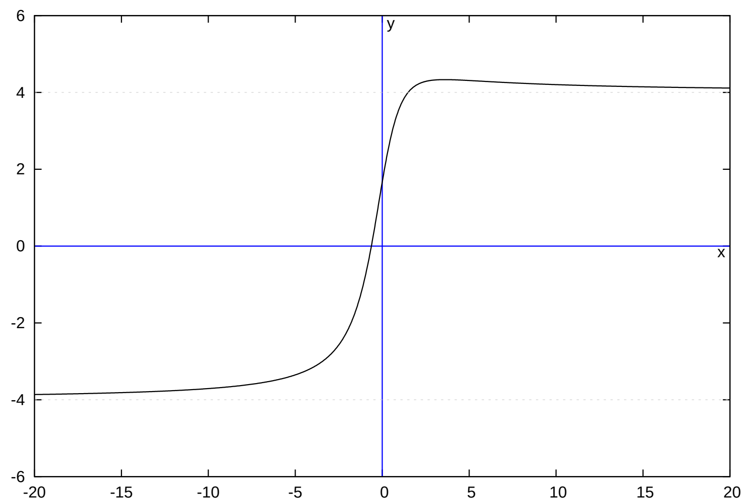
$$\lim_{x\to\infty} f(x) = a \quad \text{or} \quad \lim_{x\to-\infty} f(x) = a.$$

Example 5 Referring back to Example 3, find all horizontal asymptotes of the graph of $f(x) = \dfrac{8x+5}{\sqrt{4x^2+9}}$.

From above, we have

$$\lim_{x\to\infty} \frac{8x+5}{\sqrt{4x^2+9}} = 4 \quad \text{and} \quad \lim_{x\to-\infty} \frac{8x+5}{\sqrt{4x^2+9}} = -4.$$

The graph has two horizontal asymptotes: $y = 4$, and $y = -4$. The graph and its asymptotes are shown below.



While the graph of a function can have at most two horizontal asymptotes (as per the definition), the graph of a rational function can have only one or none. If we look back at Example 2, the following observations should be clear.

Horizontal asymptotes of rational functions

Suppose R is a rational function (i.e. a quotient of two polynomials).

- If the degree of the numerator is *less than* the degree of the denominator, then $y = 0$ is the horizontal asymptote of the graph of R.
- If the degree of the numerator is *equal to* the degree of the denominator, then $y = a/b$ is the horizontal asymptote of the graph of R, where a and b are the leading coefficients of the numerator and denominator, respectively.
- If the degree of the numerator is *greater than* the degree of the denominator, then the graph of R does not have a horizontal asymptote.