Lecture 28: L'Hôpital's rule

Objectives:

(28.1) Identify indeterminate forms.

(28.2) Apply L'Hôpital's rule to resolve indeterminate forms.

(28.3) Use algebraic techniques to rewrite limits so that L'Hôpital's rule applies.

Indeterminate forms

When evaluating limits, we have come to use the word "form" to describe the result of the substitution process. Sometimes the "form" is a number, as in

$$\lim_{x\to 5} \frac{x}{x+3} = \frac{5}{8}.$$

Sometimes the "form" is an expression that we can interpret rather easily. For example, we saw the k/∞ forms in lecture 26:

$$\lim_{x\to\infty} \frac{7}{x} = 0.$$

Other times the "form" is an expression that is ambiguous (*indeterminate*) and needs to be resolved, as in the $0/0$ indeterminate form arising in the well-known limit:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

It takes practice and experience to become good at recognizing and resolving indeterminate forms. The most common indeterminate forms are:

$$\frac{0}{0}, \quad ,\pm\frac{\infty}{\infty}, \quad \infty-\infty, \quad 0\cdot\infty, \quad 1^{\infty}, \quad 0^0, \quad \infty^0.$$

No conclusion can ever be drawn from an indeterminate form—more work is always required! We have already learned some strategies for resolving the indeterminate form $0/0$ (lecture 5) and the indeterminate form ∞/∞ (lecture 26). In this lecture, we will study L'Hôpital's rule, which will provide us with a powerful technique for resolving certain indeterminate forms.

L'Hôpital's rule

Theorem 1 — L'Hôpital's rule
Suppose f and g are differentiable functions on an open interval containing c, except possibly at c.

1. If $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)},$$

   assuming the limit on the right exists or is infinite.

2. If $\lim_{x\to c} f(x) = \pm\infty$ and $\lim_{x\to c} g(x) = \pm\infty$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)},$$

   assuming the limit on the right exists or is infinite.

These results also hold (with the obvious modifications) if we are considering one-sided limits or limits at infinity (i.e., $c = \pm\infty$).

Some comments are in order before we begin to use L'Hôpital's rule.

- The rule applies only to the indeterminate forms $0/0$ or $\pm\infty/\infty$. The rule does not apply to other indeterminate forms or if the form is not indeterminate in the first place.
- Notice that when we apply the rule, we determine the derivatives of the numerator and denominator separately; L'Hôpital's rule is not an application of the quotient rule.

Example 1 Evaluate the limit: $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$

Even though we already know the limit, let's use L'Hôpital's rule to evaluate it. Let $f(x) = \sin x$ and $g(x) = x$. Both f and g are differentiable everywhere, and the limit has a $0/0$ form. The first part of the theorem applies. It follows that

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1.$$

Example 2 Evaluate the limit: $\lim\limits_{x\to 1} \dfrac{\sin \pi x}{\ln x}$

Direct substitution results in the $0/0$ indeterminate form. Both the numerator and denominator functions are differentiable on an interval around $x = 1$. L'Hôpital's rule applies.

$$\lim_{x\to 1} \frac{\sin \pi x}{\ln x} = \lim_{x\to 1} \frac{\pi \cos \pi x}{1/x} = \lim_{x\to 1} \pi x \, \cos \pi x = (\pi)(1)(-1) = -\pi$$

Example 3 Evaluate the limit: $\lim\limits_{x\to 0} \dfrac{x - \sin x}{x^2}$

Direct substitution results in the $0/0$ indeterminate form. Both the numerator and denominator functions are differentiable on an interval around $x = 0$. L'Hôpital's rule applies.

$$\lim_{x\to 0} \frac{x - \sin x}{x^2} = \lim_{x\to 0} \frac{1 - \cos x}{2x}$$

Direct substitution into the limit on the right results in another $0/0$ form. L'Hôpital's rule applies again!

$$\lim_{x\to 0} \frac{1 - \cos x}{2x} = \lim_{x\to 0} \frac{-\sin x}{2} = 0.$$

The original limit exists and its value is 0.

Example 4 Evaluate the limit: $\lim\limits_{x\to 2} \dfrac{x^2 - 6x}{x^2}$

Direct substitution results in the number $-8/4$. The limit is -2. If we were to apply L'Hôpital's rule here, we would be wrong! The rule only applies to $0/0$ or ∞/∞.

Example 5 Evaluate the limit: $\lim\limits_{x\to \infty} \dfrac{x}{\ln x}$

Direct substitution results in the form ∞/∞. The numerator and denominator functions are differentiable for all $x > 0$. Therefore, the second part of the theorem applies with $c = \infty$.

$$\lim_{x\to \infty} \frac{x}{\ln x} = \lim_{x\to \infty} \frac{1}{1/x} = \lim_{x\to \infty} x = \infty$$

The limit is $+\infty$.

Other indeterminate forms

L'Hôpital's rule does not apply (directly) to the indeterminate forms $0\cdot\infty$, $\infty-\infty$, 1^∞, 0^0, and ∞^0. However, it is often the case that some relatively simple algebraic operations can be used to turn one form of a limit into another form. Here are some guidelines that might be useful.

- For the form $0 \cdot \infty$...
  Every limit that results in the $0 \cdot \infty$ form can be rewritten as a limit that results in $0/0$ or $\pm\infty/\infty$. To do so, rewrite the multiplication as division by the reciprocal.
- For the form $\infty - \infty$...
  These forms can usually be changed by combining terms. Write the difference of two expressions as a single expression, perhaps by getting a common denominator (if appropriate).
- For the forms 1^∞, 0^0, or ∞^0...
  Consider the limit of the logarithm of the original function. In such a case, the properties of logarithms could be used to simplify. After computing the limit of the logarithm, be sure to undo the logarithm by exponentiating.

Example 6 Evaluate the limit: $\displaystyle\lim_{x\to 0^+} x\ln x$

This limit has the form $0 \cdot (-\infty)$. Let's rewrite the multiplication as division by the reciprocal.

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x}$$

The new limit has the form $-\infty/\infty$. The numerator and denominator are differentiable functions to the right of $x = 0$, so L'Hôpital's rule applies.

$$\lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0.$$

The original limit exists and its value is 0.

Example 7 Evaluate the limit: $\displaystyle\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$

This limit has the form $\infty-\infty$. Let's use the common denominator $x\sin x$ to rewrite the function as a single fraction rather than a difference.

$$\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x\to 0^+} \left(\frac{\sin x - x}{x\sin x}\right)$$

The new limit has the form $0/0$, and now L'Hôpital's rule applies. Try to finish it on your own. You will need to use L'Hôpital's rule twice, but you should get a limit of zero.

Example 8 Evaluate the limit: $\displaystyle\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x$

This limit has the form 1^∞. Let's consider the limit of the natural logarithm of the function.

$$\lim_{x\to\infty} \ln\left(1 + \frac{1}{x}\right)^x \lim_{x\to\infty} x\ln\left(1 + \frac{1}{x}\right) = \lim_{x\to\infty} \frac{\ln(1 + 1/x)}{1/x}$$

The limit on the right has a $0/0$ form, to which L'Hôpital's rule can be applied. That limit will end up being 1 (try it!), but you must remember that 1 is the logarithm of the limit we seek. Since $\ln e = 1$, the original limit is e.