Lecture 3: Estimating limits graphically and numerically

Objectives:

(3.1) State and explain the informal definition of limit.

(3.2) Estimate limits graphically and numerically.

(3.3) State and explain, with examples, the ways limits may fail to exist.

Introductory examples

Example 1 Suppose a hot dog costs about \$1.50 (but not exactly \$1.50) and a soda costs exactly \$1.

1. Can you determine the exact cost of 2 hot dogs and 3 sodas?

   No. We can find the exact cost of the sodas, but we can only approximate the cost of the dogs.

2. About how much do 2 hot dogs and 3 sodas cost?

   The total cost will be approximately $2(1.50) + 3(1.00)$ or about \$6.00.

3. The closer the cost of the hot dog is to \$1.50, the closer the total cost is to what?

   The total cost will get closer and closer to \$6.00.

Example 2 Craig's bank charges him a fee for overdrawing his checking account. If x is the amount overdrawn, then the bank charges him $f(x)$ dollars, where

$$f(x) = \begin{cases} 20 + 0.1x, & 0 < x < 100 \\ 40 + 0.1(x - 100), & 100 \leq x < 200 \\ 60 + 0.1(x - 200), & 200 \leq x \leq 300 \end{cases}$$

1. Craig is certain he will overdraw his account by approximately \$100. About how much will he pay in fees?

   It depends. If he overdraws by less than \$100, he should expect to pay about $20 + 0.1(100)$ or \$30. If he overdraws \$100 or more, he should expect to pay about $40 + 0.1(100 - 100)$ or \$40.

2. How is this example different from the first example?

   In the first example, it didn't make any difference if the cost of the hot dog was slightly greater than or less than \$1.50. In this example, there is a jump in the fee as soon as Craig overdraws \$100.

3. The closer the amount overdrawn gets to \$100, the closer the fee gets to what?

   Well, we can't say. It depends on whether the amount overdrawn is more than \$100 or less than \$100.

Example 3 For some time, the local Burger King ran the following promotion: Each day at open, the BK manager would randomly select an amount between \$5.00 and \$15.00. If your pre-tax bill came to that amount, your meal was free. Suppose the tax rate is 7.25%, and the manager's lucky number is \$8.00.

1. If you expect to buy about \$8 worth of food, about how much should you expect to spend?

   It is unlikely that your bill will come to exactly \$8.00. You should expect to spend about $8.00 + 0.0725(8.00)$ or \$8.58.

2. The closer your food costs get to \$8, the closer your total bill will get to what?

   Well, \$8.58, just as above. The only way you'll get your meal free is if it comes to exactly \$8.00, not close to \$8.00.

3. What happens if your food costs come to exactly \$8?

   Then it's free, but you shouldn't count on this!

Informal definition of limit - Estimating limits

Informal definition of limit
Suppose the function f is defined on an open interval containing the number c, but f need not be defined at c. If $f(x)$ can be made arbitrarily close to the number L by choosing x sufficiently close to, but different from, c then we say $f(x)$ approaches L as x approaches c. We write

$$\lim_{x\to c} f(x) = L.$$

Using this new concept, we can summarize what we found in the introductory examples:

$$\lim_{x\to 1.5}(2x+3) = 6, \qquad \lim_{x\to 100} f(x) \text{ does not exist (DNE)}, \qquad \lim_{x\to 8}(x+0.0725x) = 8.58$$

Notice that a limit tells us something about the behavior of a function near a limit point, but it tells us nothing about the behavior at the point. This is a very important point!

A limit tells us about what happens near a particular point, not at the point. What the function does at the limit point is irrelevant.

We will learn how to use algebraic techniques to compute limits, but for now we'll focus on estimating limits.

Roughly speaking to estimate a limit, you should ask yourself: Are the function's y-values getting closer and closer to a particular number as the x-values get close to c (from either side)? If so, that particular number is your limit.

Example 4 Use a table of numerical values to justify $\lim_{x\to 1.5}(2x+3) = 6$.

Approaching 1.5 from the left	Approaching 1.5 from the right
$x = 1.4, \quad f(x) = 5.8$	$x = 1.6, \quad f(x) = 6.2$
$x = 1.49, \quad f(x) = 5.98$	$x = 1.51, \quad f(x) = 6.02$
$x = 1.499, \quad f(x) = 5.998$	$x = 1.501, \quad f(x) = 6.002$
$x = 1.4999, \quad f(x) = 5.9998$	$x = 1.5001, \quad f(x) = 6.0002$
$x = 1.49999, \quad f(x) = 5.99998$	$x = 1.50001, \quad f(x) = 6.00002$
$x = 1.499999, \quad f(x) = 5.999998$	$x = 1.500001, \quad f(x) = 6.000002$

Based on the numbers above, it seems reasonable to conclude that the values of $2x+3$ approach 6 as $x \to 1.5$.

Example 5 Use a table of numerical values to estimate $\lim_{x\to 2} \frac{x^2-4}{x-2}$.

Approaching 2 from the left	Approaching 2 from the right
$x = 1.9, \quad f(x) = 3.900000$	$x = 2.1, \quad f(x) = 4.100000$
$x = 1.99, \quad f(x) = 3.990000$	$x = 2.01, \quad f(x) = 4.010000$
$x = 1.999, \quad f(x) = 3.999000$	$x = 2.001, \quad f(x) = 4.001000$
$x = 1.9999, \quad f(x) = 3.999900$	$x = 2.0001, \quad f(x) = 4.000100$
$x = 1.99999, \quad f(x) = 3.999990$	$x = 2.00001, \quad f(x) = 4.000010$
$x = 1.999999, \quad f(x) = 3.999999$	$x = 2.000001, \quad f(x) = 4.000001$

Based on the numbers above, it seems reasonable to conclude that $\lim_{x\to 2} \frac{x^2-4}{x-2} = 4$.

Example 6 Use a table of numerical values to estimate $\lim_{x\to -1} \frac{|x+1|}{x+1}$.

Approaching -1 from the left	Approaching -1 from the right
$x = -1.1, \quad f(x) = -1.0$	$x = -0.9, \quad f(x) = 1.0$
$x = -1.01, \quad f(x) = -1.0$	$x = -0.99, \quad f(x) = 1,0$
$x = -1.001, \quad f(x) = -1.0$	$x = -0.999, \quad f(x) = 1.0$
$x = -1.0001, \quad f(x) = -1.0$	$x = -0.9999, \quad f(x) = 1.0$
$x = -1.00001, \quad f(x) = -1.0$	$x = -0.99999, \quad f(x) = 1.0$
$x = -1.000001, \quad f(x) = -1.0$	$x = -0.999999, \quad f(x) = 1.0$

It looks like the values of $f(x) = \dfrac{|x+1|}{x+1}$ approach two different numbers depending on how we approach $x = -1$. It seems reasonable to conclude that $\lim_{x \to -1} \dfrac{|x+1|}{x+1}$ DNE.

Example 7 Let $\chi(x)$ be the characteristic function of the rationals:

$$\chi(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

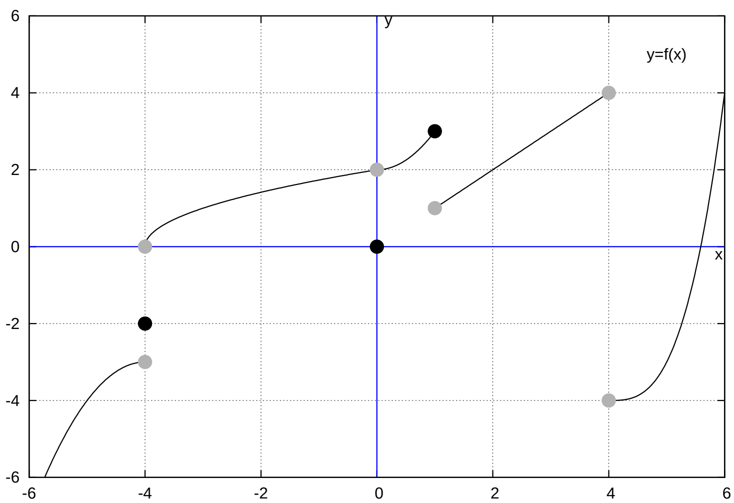
Argue that $\lim_{x \to c} \chi(x)$ DNE for any number c.

A table would not be helpful because you would probably only sample the function at rational x-values and the function would appear constant. However, since there are infinitely many rational and irrational numbers in ANY open interval containing c, the function values are always bouncing between 0 and 1. The values of $\chi(x)$ are not consistently near 0 or 1.

Well-drawn graphs can be quite useful for estimating limits. The following example leads us in that direction.

Example 8 The graph of $y = f(x)$ is shown below. Use the graph to justify each of these limits.

$$\lim_{x \to 2} f(x) = 2 \qquad \lim_{x \to 0} f(x) = 2 \qquad \lim_{x \to 4} f(x) \text{ DNE} \qquad \lim_{x \to -4} f(x) \text{ DNE}$$



Ways that limits fail to exist

In the examples above, we saw some limits that failed to exist. There are several common ways that limits can fail to exist at a point.

Failure #1 - The limit from the left does not equal the limit from the right

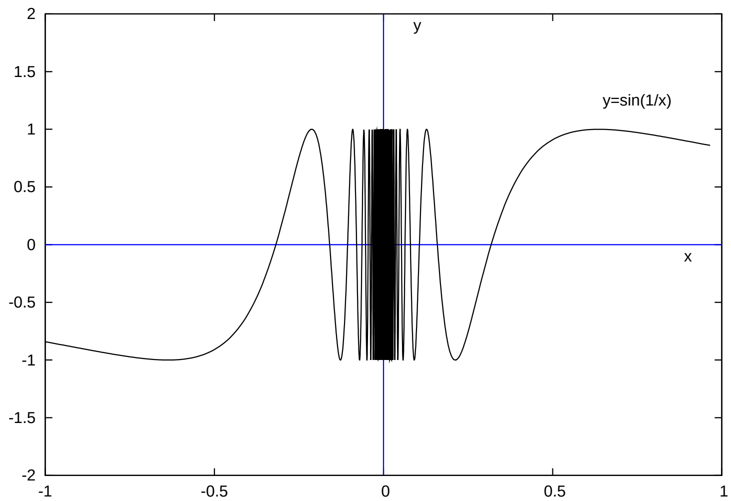
This behavior can be seen in several of the examples above. For instance, look back at Example 8 and consider the limits at $x = -4$, $x = 1$, and $x = 4$. Another excellent example of this behavior can be seen when considering $\lim_{x \to 0} \dfrac{|x|}{x}$. From the left the limit is -1, but from the right the limit is $+1$.

Failure #2 - The function values grow without bound as the limit point is approached

The examples above do not illustrate this behavior, but here is a simple example that does: $\lim_{x \to 0} \dfrac{1}{x^2}$. As x gets closer and closer to 0, the values of $f(x) = 1/x^2$ are always positive, and they grow without bound. The function values do not get closer to a limiting value. The limit DNE. Notice that the graph of $f(x) = 1/x^2$ has a vertical asymptote at $x = 0$. A limit cannot exist at a point where a function's graph has a vertical asymptote.

Failure #3 - The function values continually oscillate and approach no fixed value

Example 7 illustrates this behavior. For another example, let $f(x) = \sin(\frac{1}{x})$ and consider the limit as $x \to 0$. As x approaches 0, $\sin(1/x)$ cycles through more and more periods, continually ranging between -1 to 1. See the graph below.



Failure #4 - The function is not defined on an open interval containing the limit point

For example, $\lim_{x\to 2} \sqrt{1-x^2}$ DNE because $f(x) = \sqrt{1-x^2}$ is only defined on the interval from $x = -1$ to $x = 1$. Since we cannot even approach $x = 2$, we cannot have a limit at $x = 2$.

Example 9 As mentioned above, tables are not always helpful when trying to estimate a limit. In fact, tables can be downright misleading. For example, referring to Failure #3 (above), it should be clear that $f(x) = \sin(\pi/x)$ does not have a limit at $x = 0$. However, a typical table of values may actually suggest a limit.

Table 1	Table 2
$x = 0.1, \quad f(x) = 0$	$x = 0.3, \quad f(x) = -0.866025$
$x = 0.01, \quad f(x) = 0$	$x = 0.03, \quad f(x) = -0.866025$
$x = 0.001, \quad f(x) = 0$	$x = 0.003, \quad f(x) = -0.866025$
$x = 0.0001, \quad f(x) = 0$	$x = 0.0003, \quad f(x) = -0.866025$
$x = 0.00001, \quad f(x) = 0$	$x = 0.00003, \quad f(x) = -0.866025$
$x = 0.000001, \quad f(x) = 0$	$x = 0.000003, \quad f(x) = -0.866025$

Other sequences of x-values suggest other "possible" limits.

Some history

The modern idea of limit dates back to around 1820 and is mostly due to Augustin-Louis Cauchy. His definition went like this:

When the values successively attributed to a variable approach indefinitely to a fixed value, in a manner so as to end by differing from it by as little as one wishes, this last is called the limit of all the others.