Lecture 5: Finding limits analytically—Simple indeterminate forms

Objectives:

(5.1) Use algebraic techniques to resolve 0/0 indeterminate forms.

(5.2) Use the squeeze theorem to evaluate limits.

(5.3) Use trigonometric techniques to resolve 0/0 indeterminate forms.

Functions that differ at a point

The substitution technique that we saw in the last lecture is the simplest and most useful of all the limit techniques we will see. However, it has two major drawbacks.

The first drawback is theoretical: the substitution technique often creates very serious confusion regarding the limit concept. It is extremely important to remember that in general, a limit is not about what happens at a point, but rather what happens near the point. The behavior of the function at the limit point is irrelevant when determining a limit. Having said that, it is ironic that the substitution technique actually requires us to evaluate the function at the limit point. **You should not think about the substitution technique as function evaluation, but rather as the end result of our limit laws.**

The second drawback is practical: the substitution technique can, and often does, fail, even when a limit exists. Consider these examples:

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=4 \qquad\qquad \lim_{x\to 0}\frac{\sin x}{x}=1$$

Substitution is useless because neither expression is defined at the corresponding limit point. Nonetheless the limits exist (as we shall seen very soon). In both cases, direct substitution produces the "form" 0/0. 0/0 is not a number and it does not simplify to a number, but it is what each of the expressions looks like when we substitute the limit point for the variable. In one case, the 0/0 form is associated with the limit 4, and in the other case, the 0/0 form is associated with the limit 1. Because 0/0 forms can be resolved to give different limits (or none at all), we say that 0/0 is an indeterminate form.

> **When evaluating a limit, if substitution results in the indeterminate form 0/0, then no conclusion can be made without doing more work. The limit may or may not exist.**

We will see several ways to resolve indeterminate forms. Most of our strategies will rely on the following theorem.

> **Theorem 1 — Functions that differ at a point**
> Suppose $f(x)=g(x)$ for all $x\neq c$ on an open interval containing c. If $\lim_{x\to c} g(x)$ exists, then $\lim_{x\to c} f(x)$ also exists and
> $$\lim_{x\to c} f(x)=\lim_{x\to c} g(x).$$

To understand the value of this theorem, think about how we simplified expressions in our earlier algebra courses. For example,

$$\frac{x^2-4}{x-2}=\frac{(x-2)(x+2)}{(x-2)}=\frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}}=x+2, \qquad x\neq 2.$$

Since the expression on the far left is not defined when $x=2$, equality certainly does not hold at $x=2$, but it holds everywhere else. The functions on the left and right differ only at the single point $x=2$. According to the theorem,

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}(x+2)=4.$$

The most common applications of Theorem 1 make use of the following algebraic strategies:

1. Factor and cancel (as above)
2. Expand and simplify
3. Multiply by the conjugate and simplify
4. Get a common denominator and add (or subtract)

The following examples illustrate these strategies in order.

Example 1 Evaluate: $\lim_{x \to -2} \dfrac{x^2 - x - 6}{x^2 + 5x + 6}$

Direct substitution yields the indeterminate form 0/0. We cannot draw a conclusion without more work. Let's try factoring.

$$\lim_{x \to -2} \frac{x^2 - x - 6}{x^2 + 5x + 6} = \lim_{x \to -2} \frac{(x+2)(x-3)}{(x+2)(x+3)} = \lim_{x \to -2} \frac{\cancel{(x+2)}(x-3)}{\cancel{(x+2)}(x+3)} = \lim_{x \to -2} \frac{x-3}{x+3} = -5$$

The functions

$$f(x) = \frac{x^2 - x - 6}{x^2 + 5x + 6} \quad \text{and} \quad g(x) = \frac{x-3}{x+3}$$

are identical everywhere except at $x = -2$. By Theorem 1, they have the same limits everywhere, in particular at $x = -2$.

Example 2 Evaluate: $\lim_{h \to 0} \dfrac{(1+h)^2 - 1}{h}$

Direct substitution yields the indeterminate form 0/0. We cannot draw a conclusion without more work. Let's use the fact that $(1+h)^2 = 1 + 2h + h^2$.

$$\lim_{h \to 0} \frac{(1+h)^2 - 1}{h} = \lim_{h \to 0} \frac{2h + h^2}{h} = \lim_{h \to 0} \frac{\cancel{h}(2+h)}{\cancel{h}} = \lim_{h \to 0} (2+h) = 2$$

Is it clear how we are using Theorem 1?

Example 3 Evaluate: $\lim_{r \to 3} \dfrac{\sqrt{r+1} - 2}{r - 3}$

Direct substitution yields the indeterminate form 0/0. We cannot draw a conclusion without more work. Rationalizing the numerator will help. Since we only expect cancellation in the numerator, we will only carry out the multiplication there.

$$\lim_{r \to 3} \frac{\sqrt{r+1} - 2}{r - 3} \cdot \frac{\sqrt{r+1} + 2}{\sqrt{r+1} + 2} = \lim_{r \to 3} \frac{r + 1 - 4}{(\sqrt{r+1} + 2)(r-3)} = \lim_{r \to 3} \frac{\cancel{(r-3)}}{(\sqrt{r+1} + 2)\cancel{(r-3)}} = \frac{1}{4}$$

Is it clear how we are using Theorem 1?

Example 4 Evaluate: $\lim_{x \to 0} \dfrac{\dfrac{1}{x+4} - \dfrac{1}{4}}{x}$

Once again direct substitution yields the indeterminate form 0/0. We cannot draw a conclusion without more work. In this case, it will help to get a common denominator and carry out the subtraction in the numerator.

$$\lim_{x \to 0} \frac{\dfrac{1}{x+4} - \dfrac{1}{4}}{x} = \lim_{x \to 0} \frac{\dfrac{4 - (x+4)}{4(x+4)}}{x} = \lim_{x \to 0} \frac{-x}{4(x+4)x} = \lim_{x \to 0} \frac{-\cancel{x}}{4(x+4)\cancel{x}} = \frac{-1}{16}$$

The squeeze theorem

Theorem 1 is a comparison theorem—two functions are compared and information about one is obtained from the other. The next theorem is also a comparison theorem. It doesn't have the practical importance of Theorem 1, but it can be quite useful.

Theorem 2 — Squeeze theorem
Suppose f, g, and h are functions for which

$$g(x) \le f(x) \le h(x)$$

for all x in an open interval containing c, except possibly at c. If

$$\lim_{x\to c} g(x) = L \quad \text{and} \quad \lim_{x\to c} h(x) = L,$$

then f must have the same limit at c:

$$\lim_{x\to c} f(x) = L.$$

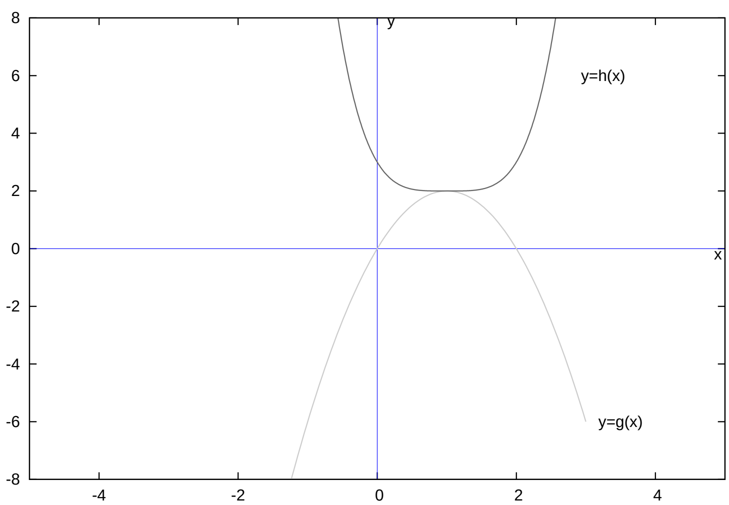
In the theorem, the values of f are squeezed between the values of g and h. If g and h are tending to a common limit, then f is going with them.

Example 5 Suppose f is a function with the property that

$$-2x^2 + 4x \le f(x) \le x^4 - 4x^3 + 6x^2 - 4x + 3$$

for all x. What can be said about $\lim_{x\to 1} f(x)$?

Let $g(x) = -2x^2 + 4x$ and $h(x) = x^4 - 4x^3 + 6x^2 - 4x + 3$. The graphs of g and h are shown below. Notice that $\lim_{x\to 1} g(x) = \lim_{x\to 1} h(x) = 2$ (by direct substitution).



Since f is "squeezed between" g and h, it follows that $\lim_{x\to 1} f(x) = 2$.

The squeeze theorem can be used to prove the following useful result.

Theorem 3 — A special limit

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Justification for Theorem 3 is given on the next page. This result is more general than it seems at first glance. You should interpret it as saying:

$$\frac{\sin g(x)}{g(x)} \to 1 \quad \text{as} \quad g(x) \to 0.$$

For example,

$$\lim_{x\to 0} \frac{\sin 5x}{5x} = 1 \qquad \lim_{x\to 1} \frac{\sin(x-1)}{x-1} = 1 \qquad \lim_{x\to 2} \frac{\sin(x^2-4)}{x^2-4} = 1$$

The next examples illustrate Theorem 3.

Example 6 Use Theorem 3 to evaluate $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x}$.

Direct substitution yields the indeterminate form 0/0. We cannot draw a conclusion without more work. Let's multiply by the conjugate of the numerator.

$$\lim_{x\to 0} \frac{1-\cos x}{x} \cdot \frac{1+\cos x}{1+\cos x} = \lim_{x\to 0} \frac{1-\cos^2 x}{x(1+\cos x)} = \lim_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)}.$$

Now use limit laws to rewrite...

$$\lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{\sin x}{1+\cos x} = 1 \cdot 0 = 0.$$

Example 7 Evaluate: $\displaystyle\lim_{x\to 0} \frac{\sin 7x}{x}$

Direct substitution yields 0/0. More work! Since $\sin 7x \neq 7 \sin x$, we cannot simplify $\sin 7x$. However, it looks like Theorem 3 should be useful. We can use it after multiplying by 7/7.

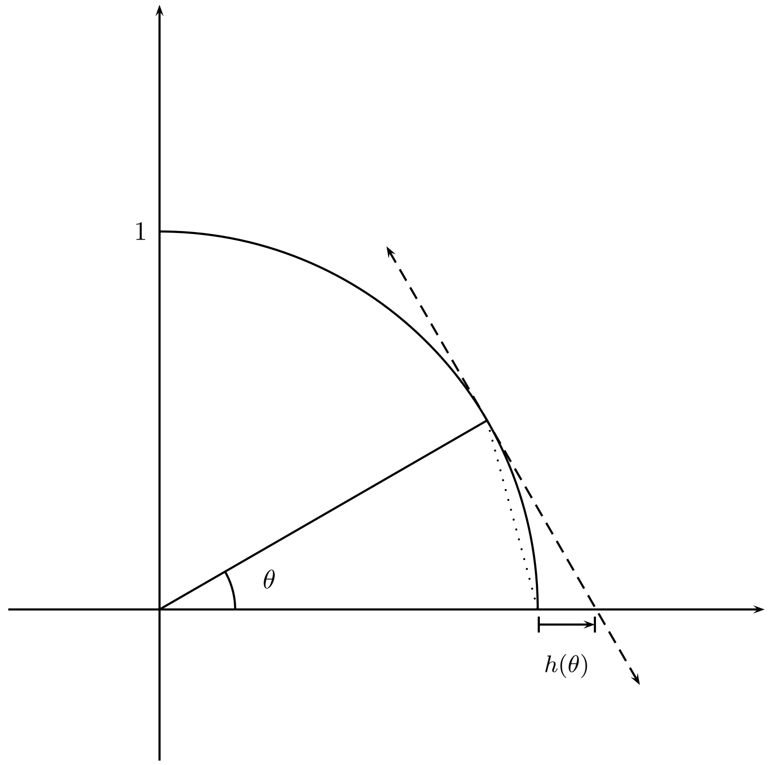
$$\lim_{x\to 0} \frac{\sin 7x}{x} = \frac{7}{7} \cdot \lim_{x\to 0} \frac{\sin 7x}{x} = 7 \cdot \lim_{x\to 0} \frac{\sin 7x}{7x} = 7 \cdot 1 = 7$$

Example 8 Evaluate: $\displaystyle\lim_{x\to 0} \frac{\tan 3x}{8x}$

Direct substitution yields 0/0. More work!

$$\lim_{x\to 0} \frac{\tan 3x}{8x} = \frac{3}{3} \cdot \lim_{x\to 0} \frac{1}{8} \frac{\sin 3x}{x \cos 3x} = \frac{3}{8} \cdot \lim_{x\to 0} \frac{\sin 3x}{3x} \frac{1}{\cos 3x} = \frac{3}{8} \cdot 1 \cdot 1 = \frac{3}{8}$$

Some justification for Theorem 3...



$$\text{Area of small triangle} \quad \leq \quad \text{Area of sector} \quad \leq \quad \text{Area of big triangle}$$

$$\frac{\sin\theta}{2} \leq \frac{\theta}{2} \leq \frac{(1+h(\theta))\sin\theta}{2} \quad \text{or} \quad 1 \leq \frac{\theta}{\sin\theta} \leq 1+h(\theta)$$

Now as $\theta \to 0$, it should be pretty clear that $h(\theta) \to 0$. Therefore we get

$$1 \leq \lim_{x\to 0} \frac{\theta}{\sin\theta} \leq 1$$

It follows that

$$\lim_{\theta\to 0} \frac{\theta}{\sin\theta} = 1$$

By the limit laws, the limit of the reciprocal function is the reciprocal of the limit:

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$